Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 12 October.

5.1 A minimum

Show that the uniform motion of a free mass point in $\mathbb{R}^3$ is a minimum of the corresponding variational problem (cf. Arnold's book).

5.2 Area-preservation

Show that a planar dynamical system (i.e. a system defined on $\mathbb{R}^2$) is Hamiltonian if and only if it is area-preserving.

5.3 Steiner ellipse

A particle P of unit mass moves in the plane of a given fixed triangle $A_1A_2A_3$. The force F_i on P is directed towards A_i and has magnitude $\Gamma\overline{PA_i}$ for $i = 1, 2, 3$, where Γ is a positive constant, and $\overline{PA_i}$ denotes the distance between P and A_i. Prove that there is a motion of P whose orbit coincides with the Steiner ellipse S of $A_1A_2A_3$ (the ellipse S passes through the vertices and the tangent at each vertex is parallel to the opposite side). Show moreover that P covers the three arcs A_1A_2, A_2A_3 and A_3A_1 of S in equal time.

5.4 Lagrange multiplier in the Neumann system

The Neumann system consists of a particle of unit mass moving on the unit sphere

$$x_0^2 \;+\; x_1^2 \;+\; \ldots \;+\; x_n^2 \;=\; 1$$

in $\mathbb{R}^{n+1}$ with a potential energy $U(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, A\mathbf{x})$ where A is a $(n+1)\times(n+1)$ diagonal matrix with eigenvalues $0 < a_0 < a_1 < \ldots < a_n$.

1. Use Hamilton's principle with a Lagrange multiplier to deduce the equations of motion and to determine the constraint force as a function of the position and velocity of the particle.

2. Show that the sphere $x_0^2 + x_1^2 + \ldots + x_n^2 = 1$ is invariant under Hamilton's equations for the Hamilton function

   $$H(\mathbf{x}, \mathbf{p}) \;=\; \frac{1}{2}(|\mathbf{p}|^2|\mathbf{x}|^2 \;-\; \langle \mathbf{x}, \mathbf{p}\rangle^2) \;+\; \frac{1}{2}\langle \mathbf{x}, A\mathbf{x}\rangle \;,$$

   by showing that the function $x_0^2 + x_1^2 + \ldots + x_n^2$ is a constant of motion. Show that the equations of motion resulting this way agree with the equations of motion in 1.