Conservative Dynamical Systems 2010/2011

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The homework consists of two exercises, to be handed in on Tuesday 26 October.

7.1 Poisson and Lie brackets

In this exercise we study some properties of Poisson and Lie brackets on symplectic manifolds.

1. Show that if two functions F, G are integrals of H then so is $\{F, G\}$.
2. Show that if $\phi : \mathcal{P} \to \mathcal{P}$ is symplectic then $\{F, G\} \circ \phi = \{F \circ \phi, G \circ \phi\}$. What is the meaning of this equation?
3. Show that the map $F \mapsto X_F$ is a Lie algebra anti-homomorphism between the Lie algebra $(C^\infty(\mathcal{P}), \{,\})$ of smooth functions $\mathcal{P} \to \mathbb{R}$ and the Lie algebra $(\mathcal{F}^\infty(\mathcal{P}), [,])$ of smooth Hamiltonian vector fields on $\mathcal{P}$. In other words, show that $[X_F, X_G] = -X_{\{F,G\}}$ where the Lie bracket $[X, Y]$ of vector fields X, Y is the vector field defined by $[X, Y](F) = X(Y(F)) - Y(X(F))$ for any function $F : \mathcal{P} \to \mathbb{R}$.

7.2 The orthogonal group $O(n, \mathbb{R})$

Let $gl(n, \mathbb{R})$ be the set of all real $n \times n$–matrices. Further define

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\begin{aligned}
Gl(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid \det S \neq 0\} \\
O(n,\mathbb{R}) &= \{S \in gl(n,\mathbb{R}) \mid S^T S = \mathrm{id}\} \\
o(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^T = -A\} \\
Sym(n,\mathbb{R}) &= \{A \in gl(n,\mathbb{R}) \mid A^T = A\}.
\end{aligned}
$$

1. Show that $o(n, \mathbb{R})$ and $Sym(n, \mathbb{R})$ are real vector spaces. Give their dimensions.
2. Show that $gl(n, \mathbb{R})$ and $o(n, \mathbb{R})$ form Lie algebras with respect to the product given by the commutator.
3. Show that $Gl(n, \mathbb{R})$ is an n^2–dimensional manifold. Show how $gl(n, \mathbb{R})$ can be regarded as the tangent space $T_{\mathrm{id}} Gl(n, \mathbb{R})$.
4. Let $F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ be defined by $F(S) = S^T S$. Show that F is a smooth map and that the image of F is a subset of $Sym(n, \mathbb{R})$.
5. Show that the derivative[1] $D_{\mathrm{id}} F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ is given by $D_{\mathrm{id}} F(B) = B^T + B$. What is the rank of this derivative?
6. Show that the rank of the derivative $D_S F : gl(n, \mathbb{R}) \longrightarrow gl(n, \mathbb{R})$ is independent of $S \in O(n, \mathbb{R})$. *Hint:* Use the fact that $O(n, \mathbb{R})$ is a group.
7. Show that $O(n, \mathbb{R})$ is a manifold and determine its dimension. In what sense can $o(m, \mathbb{R})$ be regarded as the tangent space $T_{\mathrm{id}} O(n, \mathbb{R})$?

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[1] In another notation, $D_{\mathrm{id}} F = F_{*,\mathrm{id}}$.