Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 15 November.

10.1 Motion of a charge in an electromagnetic field

Consider a particle of mass m and charge e moving in $\mathbb{R}^3$ under the influence of a magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ and an electric field $\mathbf{E} = (E_x, E_y, E_z)$. The electric field is conservative and can thus be written as $\mathbf{E} = -\nabla\phi$ where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is called the *electric potential*. From physics we know that Newton's equations of motion for the particle are

$$m\ddot{\mathbf{r}} = e\,\dot{\mathbf{r}} \times \mathbf{B} + e\,\mathbf{E}\,.$$

Define the symplectic 2-form

$$\omega = \mathrm{d}p_x \wedge \mathrm{d}x + \mathrm{d}p_y \wedge \mathrm{d}y + \mathrm{d}p_z \wedge \mathrm{d}z + e\big(B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y\big)$$

and the Hamilton function $H(\mathbf{p}, \mathbf{r}) = (p_x^2 + p_y^2 + p_z^2)/2m + e\phi(\mathbf{r})$. Show that the Hamiltonian vector field corresponding to H on the symplectic manifold $(T^\star\mathbb{R}^3, \omega)$ give the above Newton equations of motion for a charged particle.

10.2 The Morse oscillator

Consider the Morse oscillator described by the Hamilton function

$$H = \frac{p^2}{2m} + D_e\big(\mathrm{e}^{-2aq} - 2\mathrm{e}^{-aq}\big)\,,$$

where D_e and a are positive constants, and $(p, q) \in \mathbb{R}^2$.

1. A Morse oscillator is often used to describe a chemical bond. What is the meaning of D_e in this case?

2. Show that there are two critical energies, $E_1 < E_2$, such that the level sets

   $$M_E = \{(p, q) \in \mathbb{R}^2 \,:\, H(p, q) = E\}$$

   are empty if $E < E_1$, topological circles if $E_1 < E < E_2$, and topological lines for $E > E_2$. Plot the level sets for the energies E_1, E_2, an energy between E_1 and E_2, and an energy greater than E_2.

3. For $E_1 < E < E_2$, compute the area, $A(E)$, of the region enclosed by the level set M_E in the (p, q) plane. For such energies, compute the period

$$T(E) = \frac{\mathrm{d}A(E)}{\mathrm{d}E} \,.$$

Sketch and interpret the graph of the period $T(E)$ for energies $E_1 < E < E_2$.

10.3 Reduced Euler top

Analyse the dynamics of the "reduced Euler top"

$$H(x,y,z) \quad = \quad \frac{x^2}{2a} + \frac{y^2}{2b} + \frac{z^2}{2c} \,, \qquad 0 < a \leq b \leq c$$

on $\mathbb{S}^2$ in the limiting cases $a \to b$ and $b \to c$.

10.4 Another Hamiltonian system on $\mathbb{S}^2$

In this exercise we study another Hamiltonian system defined on $\mathbb{S}^2$ which is not a cotangent bundle.

In $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) consider the submanifold $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \,:\, x^2 = 1\}$ and the 2-form

$$\omega = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2.$$

1. Show that ω is not closed. Show that ω is degenerate in $\mathbb{R}^3$ by finding at each point $x \in \mathbb{R}^3$ the space $N_x = \{\xi \in T_x\mathbb{R}^3 \,:\, \omega(\xi, -) = 0\}$. Show that the restriction $\varpi = \omega|_{\mathbb{S}^2}$ of ω to $\mathbb{S}^2$ is closed and non-degenerate. Show that ϖ is not exact.

2. Let $H : \mathbb{S}^2 \to \mathbb{R}$. Find the Hamiltonian vector field X_H on $\mathbb{S}^2$ that satisfies $\varpi(X_H, -) = dH(-)$.

3. Compute the Poisson brackets $\{x_i, x_j\}$, $i, j = 1, 2, 3$ with respect to ϖ and then compute the Poisson bracket $\{F, G\}$ for two arbitrary functions $F, G : \mathbb{S}^2 \to \mathbb{R}$.

4. Describe the dynamics of the Hamiltonian function $H = x_1$ on $\mathbb{S}^2$.

5. Show that every locally Hamiltonian vector field X on $\mathbb{S}^2$ is globally Hamiltonian, given that every boundaryless 1-chain on $\mathbb{S}^2$ is the boundary of a 2-chain.

Hint. Use appropriate coordinates on $\mathbb{S}^2$ (but be careful!).