Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 10 February.

1.1 Hamiltonian phase portraits

Rewrite the second order system $\ddot{x} = -\frac{\partial V}{\partial x}$ as a first order (Hamiltonian) system, with $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of the form $H(x,p) = \frac{1}{2}p^2 + V(x)$. For several shapes of the graph of V, sketch the phase portrait of X_H. How do the integral curves intersect the x–axis? In particular consider cases where V has maxima, minima or a horizontal asymptote.

1.2 Planar vector fields

Discuss the following list of planar vector fields. Consider the question whether they are Hamiltonian, gradient or neither of the two. Determine Hamilton or potential functions if possible. What is the connection with *exact* differential equations?

$$\begin{aligned} \dot{x} = y,\ \dot{y} = -x; &\qquad \dot{x} = -y^2,\ \dot{y} = -2xy; \\ \dot{x} = \sin y,\ \dot{y} = \cos x; &\qquad \dot{x} = 2y,\ \dot{y} = 3x; \\ \dot{x} = y - \varepsilon x,\ \dot{y} = -x - \varepsilon y; &\qquad \dot{x} = x,\ \dot{y} = -y. \end{aligned}$$

If you feel like it, you can attack these examples with all the machinery you remember from the course(s) on ODE's: determination of equilibria (stationary solutions) and the corresponding linear parts, a stability analysis, Lyapunov functions, etc.. Also it is useful to draw phase portraits.

1.3 Gradient- and Hamiltonian vector fields

Consider a smooth 1–dimensional potential $V : \mathbb{R} \longrightarrow \mathbb{R}$ and define the Hamiltonian function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by means of $H(x,p) = \frac{1}{2}p^2 + V(x)$. Consider the corresponding Hamiltonian vector field $X_H = \frac{\partial H}{\partial y}\mathrm{e}_1 - \frac{\partial H}{\partial x}\mathrm{e}_2$ and the gradient vector field $\nabla H = \frac{\partial H}{\partial x}\mathrm{e}_1 + \frac{\partial H}{\partial y}\mathrm{e}_2$. Prove that, in each point of $\mathbb{R}^2$, the integral curves of X_H and ∇H are orthogonal. What is the relation to the level curves of H ? Discuss the changes in the phase portraits of X_H and ∇H if H is replaced by $-H$. In particular consider a neighbourhood of a minimum and a saddle point of H.

1.4 Period and area

Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function given by $H(x,p) = \frac{1}{2}p^2 + V(x)$ and assume that for $z_0 \in \mathbb{R}$ the motion in the level $H^{-1}(z_0)$ is periodic. Then show that, for z near z_0 the motion is periodic as well. Let $A(z)$ denote the area enclosed by the level $H^{-1}(z)$, while $T(z)$ denotes the period of oscillation in this level set. Show that $T = \frac{\mathrm{d}A}{\mathrm{d}z}$.