Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 24 February.

3.1 Kepler's 3rd law

The 3rd law of Kepler states that the ratio of the square of the period of an elliptic orbit and the cube of its semi major axis is constant. The aim of this exercise is to check the relation to the $\frac{1}{r}$–form of the gravitational potential, from which one obtaines an effective potential after reduction to one degree of freedom.

1. Check that the minimum of the effective potential leads to a circular orbit which satisfies Kepler's 3rd law.
2. Derive the $\frac{1}{r}$–form of the gravitational potential from Kepler's 3rd law.

3.2 Linear Hamiltonian systems

What are the possible types of linear Hamiltonian systems with one degree of freedom? How do the corresponding Hamiltonian functions look like? Which of the occurring equilibria are structurally stable — and in what sense?

3.3 Collision time

Two particles P_i with mass m_i, $(i = 1, 2)$ attract each other according to Newton's law with attraction constant Γ. In the initial position, they are at rest and their distance is $2a$. When do they meet?

3.4 A bead on a wire, Huygens's isochronous curve

A bead with unit mass moves along a stiff wire, without friction. The wire lies in a vertical plane, the acceleration of gravity equals 1. Suppose the wire is given by the equation $y = U(x)$. Show that the system has energy $H = [1 + (\frac{\mathrm{d}U}{\mathrm{d}x})^2] \cdot \frac{\dot{x}^2}{2} + U(x)$. Let q be an arclength parameter along the wire and put $p := \dot{q}$. Show that the system has the Hamiltonian form $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$. Next assume that U attains a minimum at $x = x_0$. Prove that the frequency of 'small oscillations' at $x = x_0$ is equal to $\sqrt{U''(x_0)}$.

Subsequently we consider the special case where U is a cycloid, parametrically given by $\vartheta \mapsto (x, y) = (a(2\vartheta + \sin 2\vartheta), a(1 - \cos 2\vartheta))$. Here $a > 0$ is a constant while ϑ varies over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Compute its arclength parameter and obtain its equations of motion. Prove that this device implements the harmonic oscillator.