Dynamical Systems 2007

This one extended exercise is homework, to be handed in on 12 March.

5.1 Geodesics on a surface of revolution

Let r, φ and z be cylindrical coordinates on $\mathbb{R}^3 = \{x, y, z\}$: so where $x = r\cos\varphi$ and $y = r\sin\varphi$. In the (x, z)-plane a parametrized curve $x = f(v), z = g(v)$ is given, where v varies over an open interval; we assume that here always $f(v) > 0$. Without limitation of generality we also assume that $(f'(v))^2 + (g'(v))^2 = 1$, which expresses that v is an arclength parameter. This curve is revolved around the z-axis, yielding the surface $\mathcal{S}$

$$x = f(v)\cos\varphi, y = f(v)\sin\varphi, z = g(v),$$

parametrized by v and φ. We now investigate when a curve $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ is a geodesic. By definition the curve $\mathbf{R}$ is a geodesic if for all t

$$\ddot{\mathbf{R}}(t) \perp \mathcal{S}.$$

Comment. In the mechanical interpretation we look at a 'free particle' (a point mass of mass 1) moving over $\mathcal{S}$, i.e., without external forces like gravity. According to the d'Alembert principle, the point mass is kept on the surface $\mathcal{S}$ by the perpendicular force $\ddot{\mathbf{R}}(t)$.

1. Show that for a geodesic $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ one has

$$\begin{aligned} \dot{\mathbf{R}} &= \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi + \dot{z}\mathbf{e}_z \\ \ddot{\mathbf{R}} &= (\ddot{r} - r\dot{\varphi}^2)\mathbf{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi + \ddot{z}\mathbf{e}_z. \end{aligned}$$

2. Show that $r^2\dot{\varphi}$ and $\frac{1}{2}\langle\dot{\mathbf{R}}\dot{\mathbf{R}}\rangle = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)$ are two (first) integrals of the system and that moreover

$$f'\ddot{r} + g'\ddot{z} - f'r\dot{\varphi}^2 = 0.$$

   From now on we write $r(t) = f(v(t)), z(t) = g(v(t))$.

3. Show that the statements in item 2 are equivalent to
$$\begin{aligned} 2ff'\dot{v}\dot{\varphi} + f^2\ddot{\varphi} &= 0 \\ \ddot{v} - ff'\dot{\varphi}^2 &= 0. \end{aligned}$$

4. Show that from 3, in reverse, it follows that $\ddot{\mathbf{R}}(t) \perp \mathcal{S}$.

5. Define q_1, q_2, p_1 and p_2 by
$$q_1 = v, q_2 = \varphi, p_1 = \dot{v}, p_2 = f^2(v)\dot{\varphi}$$
and express $H = \frac{1}{2}\langle \dot{\mathbf{R}}\dot{\mathbf{R}} \rangle$ in q_1, q_2, p_1 and p_2. Show that 3 is equivalent to the canonical form
$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i} \ (i = 1, 2).$$
Now reinterprete the conservation laws found under 2.

6. Let $\theta = \theta(t)$ be the angle that the geodesic makes with the 'meridian'. Show that $|f\dot{\varphi}| = |\dot{\mathbf{R}}| \sin\theta$. Next show that
$$C = f \sin\theta$$
is another (first) integral; this is the celebrated theorem of Clairaut.

7. Show that all meridians of $\mathcal{S}$ are geodesics and that a parallel circle $v = v_0$ of $\mathcal{S}$ is a geodesic precisely when $f'(v_0) = 0$.

8. Fix $p_2 = M$, taking $M \neq 0$. Reduce to 1 degree of freedom by the effective potential $V_M(q_1) = M^2/(2f^2(q_1))$ (compare with the case of the central force field).
   (a) Show that if v_0 is a critical point, then the reduced system has an equilibrium $(q_1, p_1) = (v_0, 0)$. Compare with 7.
   (b) Describe the dynamics of the reduced system near such equilibria in the cases where v_0 is a maximum or a minimum.
   (c) Reinterprete the above findings for the original, unreduced system. Here describe the phase space and its decomposition in invariant level sets $p_2 = M, H = E$. What is the geometry of these sets and what is the corresponding dynamics? Also interprete the findings in the configuration space. Why is this description incomplete?

9. Explain the relationship of the items 1 - 5 with the calculus of variations.