Dynamical Systems 2007

The last two exercises are homework, to be handed in on 2 April.

8.1 A special harmonic motion

In Euclidean 3-space, the lines ℓ_i $(i = 1, 2, 3)$ are given, all three passing through the point O. The angle between any of the two lines is α $(0 < \alpha < \pi/2)$. Three particles P_i of unit mass move along the lines ℓ_i, respectively. Any two particles P_i, P_j $(i \neq j)$ attract each other by the force $k^2\overline{P_iP_j}$, where k is a positive constant and $\overline{P_iP_j}$ denotes the distance between P_i and P_j. Determine the general motion of the three particles.

8.2 The harmonic oscillator

For $a,\ b > 0$ consider the Hamiltonian function

$$H(x, y) \quad = \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \ .$$

Give the frequency of the oscillation in the level $H^{-1}(z)$, for $z > 0$. Also determine the eigenvalues of the linear part in the equilibrium. Next construct action-angle variables (I, φ) and determine both $H = H(I)$ and the vector field in terms of (I, φ). (Hint: See the next exercise but use elliptic polar coordinates.) Then define $\xi := \sqrt{2I}\cos\varphi$ and $\eta := \sqrt{2I}\sin\varphi$, and rewrite everything in the (ξ, η)-variables. What has changed since the beginning? Give a geometric interpretation, referring to the next exercise.

8.3 Hamiltonian polar coordinates

Introduce polar coordinates on the plane, by the usual formulae $x = r\cos\varphi,\ y = r\sin\varphi$. Also define $I := \frac{1}{2}r^2$. First show that $dx \wedge dy = dI \wedge d\varphi$. Now suppose, that $H = H(x, y)$ is a smooth Hamiltonian function on the plane, such that (I, φ) is a set of action-angle variables for it. Describe the general form of H. Also give the frequency of the oscillation in the level $H^{-1}(z)$.

8.4 Small oscillations

In a plane Π, a fixed homogeneous rod (length $2a$, mass density s, midpoint O) is given. A particle P of mass M moves in Π. It is attracted by any mass element dm at a point Q on the rod by the force $kMr^\alpha\, dm$. Here, k is a positive constant, $\overline{PQ} = r$ and $\alpha = 2n - 1$ for $n = 1, 2, 3, \ldots$. Obviously O is a stable equilibrium point. Determine the frequencies of small oscillations of P in the neighbourhood of O.