Exercise 9. 4. 2015

1. The aim of this exercise is to prove a Poincaré–Birkhoff fixed point theorem. Consider the annulus $A := [1,2]\times\mathbb{S}^1$, with coordinates (I,φ), where φ is counted mod 2π. Consider a smooth, boundary preserving diffeomorphism $T_\varepsilon : A \longrightarrow A$, of the form $T_\varepsilon : (I,\varphi) \mapsto (I, \varphi+2\pi\rho(I)) + \varepsilon\,(f(I,\varphi,\varepsilon), g(I,\varphi,\varepsilon))$ and such that

   - $\rho'(I) \neq 0$, saying that T_ε is a twist-map (for simplicitiy we take ρ increasing);
   - $\oint_\gamma I\,\mathrm{d}\varphi = \oint_{T_\varepsilon(\gamma)} I\,\mathrm{d}\varphi$, which means that T_ε is preserving area.

   Show that for each rational number p/q, with

   $$\rho(1) \leq \frac{p}{q} \leq \rho(2) \ ,$$

   in A there exists a periodic point of T_ε, of period q, provided that $|\varepsilon|$ is sufficiently small. *Hint:* Abbreviating $T_\varepsilon^q(I,\varphi) = (I + O(\varepsilon), \Phi_{q,\varepsilon}(I,\varepsilon))$, with $\Phi_{q,\varepsilon}(I,\varphi) = \varphi + 2\pi q\rho(I) + O(\varepsilon)$, consider the equation $\Phi_{q,\varepsilon}(I,\varphi) = \varphi + 2\pi p$, for $p \in \mathbb{Z}$. Use the implicit function theorem in order to obtain a curve $C = \{I = F(\varphi,\varepsilon)\}$ of solutions. Then study the intersection of C and $T_\varepsilon^q(C)$.