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## Exercise Sheet 1

### Action variables and periods of 1-degree-of-freedom systems.

Return by Monday, 20th April

1. Consider the harmonic oscillator with Hamilton function

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

on the phase space $(p, q) \in \mathbb{R}^2$.

   (a) For $E > 0$, plot the level sets

$$M_E = \{(p, q) \in \mathbb{R}^2 \,:\, H(p, q) = E\} \tag{1}$$

   and compute the area, $A(E)$, of the region in the $(p, q)$ plane enclosed by the level set $M_E$.

   (b) Show that

$$T(E) := \frac{\mathrm{d}A(E)}{\mathrm{d}E}\,,$$

   is the time (the period) which it takes the harmonic oscillator at energy $E$ to trace the full level set $M_E$ exactly one time.

2. Consider the Morse oscillator described by the Hamilton function

$$H = \frac{p^2}{2m} + D_e\big(\mathrm{e}^{-2aq} - 2\mathrm{e}^{-aq}\big)\,,$$

where $D_e$ and $a$ are positive constants, and $(p, q) \in \mathbb{R}^2$.

   (a) A Morse oscillator is often used to describe a chemical bond. What is the meaning of $D_e$ in this case?

   (b) Show that there are two critical energies, $E_1 < E_2$, such that the level sets $M_E$ defined as in (1) are empty if $E < E_1$, topological circles if $E_1 < E < E_2$, and topological lines for $E > E_2$. Plot the level sets for the energies $E_1$, $E_2$, an energy between $E_1$ and $E_2$, and an energy greater than $E_2$.

   (c) For $E_1 < E < E_2$, compute the area, $A(E)$, of the region enclosed by the level set $M_E$ in the $(p, q)$ plane. For such energies, compute the period

$$T(E) = \frac{\mathrm{d}A(E)}{\mathrm{d}E}\,.$$

3. Consider the pendulum with Hamilton function

$$H = \frac{p^2}{2m} - mgl\cos q \quad (g, l > 0)$$

with $(p, q) \in \mathbb{R} \times S^1 := \mathbb{R} \times \big(\mathbb{R}/(2\pi\mathbb{Z})\big)$.

(a) Show that there are two critical energies, $E_1 < E_2$, such that the level set

$$M_E = \{(p,q) \in \mathbb{R} \times \big(\mathbb{R}/(2\pi\mathbb{Z})\big) \ : \ H(p,q) = E\}$$

is empty if $E < E_1$, consists of one topological circle if $E_1 < E < E_2$, and two topological circles if $E > E_2$. Plot the level sets $M_E$ for the energies $E_1$, $E_2$, an energy between $E_1$ and $E_2$, and an energy greater than $E_2$.

(b) For $E_1 < E < E_2$ compute the area, $A_{\mathrm{osc}}(E)$, enclosed by the level set $M_E$. For $E > E_2$, compute the area, $A_{\mathrm{rot}}(E)$ enclosed by either of the lines in the level set $M_E$ and the axis $p = 0$. Note that the integrals involved in the computation of $A_{\mathrm{osc}}(E)$ and $A_{\mathrm{rot}}(E)$ are elliptic integrals which you can look up in an integral table (see, e.g., the book *Handbook of Mathematical Functions* by Abramowitz and Stegun which is available online under the URL www.math.hkbu.edu.hk/support/aands/toc.htm). Use a program like Maple or Mathematica to plot $A_{\mathrm{osc/rot}}(E)$ versus $E$.

(c) Determine the periods $T_{\mathrm{osc/rot}}(E) = \mathrm{d}A_{\mathrm{osc/rot}}(E)/\mathrm{d}E$. Again use a program like Maple or Mathematica to plot $T_{\mathrm{osc/rot}}(E)$ versus $E$. Use the asymptotics of elliptic integrals to show that for $E \to E_1$, $A_{\mathrm{osc}}(E)$ behaves like the corresponding function of an harmonic oscillator. Similarly study $A_{\mathrm{rot}}(E)$ for $E \to \infty$.