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Exercise Sheet 3

Spherical Pendulum and Monodromy.

Return by 11^{th} May

The *spherical pendulum* consists of a particle of mass m moving on the twodimensional sphere $x^2 + y^2 + z^2 = r^2$ under the influence of a gravitational force acting in the negative z direction.

1. Use spherical coordinates $(x, y, z) = r(\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$, $(\varphi, \vartheta) \in [0, 2\pi] \times [0, \pi]$, to write down the Lagrange function for the spherical pendulum.

2. Show that, after a suitable scaling, the Hamilton function in spherical coordinates and the conjugate momenta assumes the form
$$H = \frac{1}{2} p_\vartheta^2 + \frac{1}{2} \frac{p_\varphi^2}{\sin^2\vartheta} + 1 + \cos\vartheta \,.$$

3. Note that
$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \,,$$
and hence p_φ is a constant of the motion. The spherical pendulum is thus integrable. For a fixed value $p_\varphi \neq 0$, consider the effective potential for the ϑ motion,
$$V_{\text{eff}}(\vartheta) = \frac{1}{2} \frac{p_\varphi^2}{\sin^2\vartheta} + 1 + \cos\vartheta \,.$$
Argue that, for given $p_\varphi \neq 0$, there is one critical energy $E_0(p_\varphi)$ below which there is no ϑ motion, and above which the ϑ motion is oscillatory. What are the corresponding motions of the spherical pendulum for $E = E_0(p_\varphi)$ and $E > E_0(p_\varphi)$? For $p_\varphi = 0$, there are two critical energy values $E_1 = 0$ and $E_2 = 2$ (you do not need to show this). What are the corresponding motions of the spherical pendulum? Plot the critical (p_φ, E) pairs $(p_\varphi, E) = (p_\varphi, E_0(p_\varphi))$, $p_\varphi \neq 0$, $(p_\varphi, E) = (0, E_1)$ and $(p_\varphi, E) = (0, E_2)$ in the half plane $(p_\varphi, E) \in \mathbb{R} \times [0, \infty)$.

4. According to 3. the spherical pendulum is integrable and (most of) its phase space is foliated by invariant 2-tori. The spherical coordinates (and the conjugate momenta) define fundamental paths γ_φ and γ_ϑ on these 2-tori. The action associated with γ_φ is given by
$$I_\varphi = \frac{1}{2\pi} \oint_{\gamma_\varphi} p_\varphi \, \mathrm{d}\varphi = p_\varphi \,.$$
Note that I_φ can assume positive and negative values: $I_\varphi \in \mathbb{R}$. Compute the elliptic integral which gives the action associated with γ_ϑ,
$$I_\vartheta = \frac{1}{2\pi} \oint_{\gamma_\vartheta} p_\vartheta \, \mathrm{d}\vartheta \,,$$
as a function of p_φ and energy E. To this end substitute $z = \cos\vartheta$ in the integral for I_ϑ which gives
$$I_\vartheta = \frac{2}{\pi} \int_{z_1}^{z_2} \frac{f(z)}{g(z)} \frac{1}{\sqrt{2f(z)}} \, \mathrm{d}\vartheta \,,$$

where

$$f(z) := (1-z^2)(E-1-z) - \frac{1}{2}p_\varphi^2\,, \quad g(z) = 1-z^2\,,$$

and $z_1 = c$ and $z_2 = b$ are the two lower roots of the three roots $a > b > c$ of $f(z)$. It is helpful to consider the partial fraction decomposition

$$\frac{f(z)}{g(z)} = -z + h - 1 - \frac{1}{4}p_\varphi^2\left(\frac{1}{1-z} + \frac{1}{1+z}\right)\,.$$

I_ϑ is then given by a linear combination of first, second and third kind complete elliptic integrals which you can look up in an integral table. It is best to write these integrals in terms of the roots $a > b > c$ of $f(z)$.

5. For fixed energies $E > 0$, plot the action I_ϑ as a function of $p_\varphi = I_\varphi \in \mathbb{R}$. This is a surface of constant energy in the space of the actions $(I_\varphi, I_\vartheta) \in \mathbb{R} \times [0,\infty)$. Note that the frequency vector $(\omega_\varphi, \omega_\vartheta)$ is orthogonal to the energy surface in the space of the actions (I_φ, I_ϑ), and the winding number
$$W := \frac{\omega_\varphi}{\omega_\vartheta} = -\left.\frac{\partial I_\vartheta}{\partial I_\varphi}\right|_E$$
is the slope of the energy surface in action space. Note that the energy surfaces have a kink at $I_\varphi = 0$.

6. Define a 'smooth' action
$$\tilde{I}_\vartheta(p_\varphi, E) = \begin{cases} I_\vartheta(p_\varphi, E)\,, & p_\varphi \geq 0 \\ I_\vartheta(p_\varphi, E) + n(E)p_\varphi\,, & p_\varphi < 0 \end{cases}\,,$$
where $n(E) = -1$ if $0 < E < 2$ and $n(E) = -2$ if $2 < E$. For different values of E, plot $\tilde{I}_\vartheta(p_\varphi, E)$ as a function of p_φ. Note that (as opposed to the energy surfaces in 5.) the energy surfaces in the space of actions $(I_\varphi, \tilde{I}_\vartheta)$ do not have a kink at $I_\varphi = 0$.

7. Due to the isolated singular value $(p_\varphi, E) = (0,2)$ the action variables are multivalued functions of (p_φ, E). We can uncover this effect which is called *monodromy* as follows. Consider a (small) circle γ in the (p_φ, E) plane which has the critical value $(p_\varphi, E) = (0,2)$ at the center. Consider a point $P^+ = (p_\varphi^+, E^+) \in \gamma$ with $p_\varphi^+ > 0$ and the corresponding actions $(I_\varphi^+, I_\vartheta^+) = (p_\varphi^+, I_\vartheta(p_\varphi^+, E^+))$ with I_ϑ computed as in 4. Consider the point $P^- = (p_\varphi^-, E^-) \in \gamma$ with $p_\varphi^- = -p_\varphi^+ < 0$ and $E^- = E^+$. Use the construction in 6. to smoothly continue a pair of actions with initial values $(I_\varphi^+, I_\vartheta^+)$ at P^+ along γ to two pairs of actions $(I_\varphi^{-,<}, I_\vartheta^{-,<})$ and $(I_\varphi^{-,>}, I_\vartheta^{-,>})$ at P^- by connecting P^+ and P^- along the lower arc segment $\gamma_<$ and the upper arc segment $\gamma_>$ of γ which cross the line $p_\varphi = 0$ with $E < 2$ and $E > 2$, respectively. Determine the 2×2 monodromy matrix M which maps $(I_\varphi^{-,<}, I_\vartheta^{-,<})$ to $(I_\varphi^{-,>}, I_\vartheta^{-,>})$. Note that M is different from the identity matrix which means that the monodromy associated with the path γ is nontrivial.