Exercises 14. 11. 2012

1. Sketch the set

$$\Gamma_{\gamma,\tau} \quad := \quad \left\{ \omega \in \mathbb{R}^2 \;\middle|\; |2\pi\langle k \mid \omega\rangle| \geq \frac{\gamma}{|k|^\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^2 \right\}$$

of (γ, τ)–Diophantine frequency vectors, where $\gamma > 0$, $\tau > 1$ and $|k| := |k_1| + |k_2|$. What changes if $k \neq 0$ varies in $\mathbb{N}_0^2$ instead of $\mathbb{Z}^2$?

2. Prove the Lemma of Paley–Wiener : a periodic function with Fourier series $f(x) = \sum_{k\in\mathbb{Z}} f_k \, e^{2\pi i k x}$ is analytic if and only if the coefficients decay exponentially fast :

$$\bigvee_{M,\eta>0} \quad \bigwedge_{k\in\mathbb{Z}} \quad |f_k| \;\leq\; M \cdot e^{-|k|\cdot\eta} \;.$$

3. Let $\sigma \in \mathbb{R}^n$ and $S \in GL_n(\mathbb{R})$. Show that $x \mapsto Sx + \sigma$ defines an orientation preserving diffeomorphism on $\mathbb{T}^n$ if and only if $S \in SL_n(\mathbb{Z})$. Conclude that the frequency vector of a conditionally periodic torus is well defined up to the lattice $\mathbb{Z}^n$. What does this mean for vector fields X and $\Phi_* X$ on $\mathbb{T}^n$ if $\Phi - \mathrm{id}$ is sufficiently small ?