# Quasi-periodic bifurcation theory 2016/2017

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## 1 Diophantine conditions

Diophantine conditions can be expressed in various ways and the aim of this exercise is to show that these are equivalent. The Euclidean inner product of two vectors $u, v \in \mathbb{R}^n$ is

$$\langle u \mid v \rangle \quad = \quad \sum_{j=1}^{n} u_j v_j$$

and for integer vectors $k \in \mathbb{Z}^n$ we use the length

$$|k| \quad = \quad \sum_{j=1}^{n} |k_j| \ .$$

Given $\gamma > 0$ and $\tau > n$ define the sets

$$\begin{aligned}
D_{\tau,\gamma} \quad &:= \quad \left\{ \omega \in \mathbb{R}^{n+1} \ \middle| \ |\langle k \mid \omega \rangle| \geq \frac{\gamma}{|k|^\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^{n+1} \right\} \\
E_{\tau,\gamma} \quad &:= \quad \left\{ \alpha \in \mathbb{R}^{n} \ \middle| \ |\langle k \mid \alpha \rangle - \ell| \geq \frac{\gamma}{(|k| + |\ell|)^\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^{n}, \, \ell \in \mathbb{Z} \right\} \\
F_{\tau,\gamma} \quad &:= \quad \left\{ \alpha \in \mathbb{R}^{n} \ \middle| \ |\langle k \mid \alpha \rangle - \ell| \geq \frac{\gamma}{|k|^\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^{n}, \, \ell \in \mathbb{Z} \right\} \\
G_{\tau,\gamma} \quad &:= \quad \left\{ \alpha \in \mathbb{R}^{n} \ \middle| \ |\mathrm{e}^{2\pi \mathrm{i} \langle k | \alpha \rangle} - 1| \geq \frac{\gamma}{|k|^\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^{n}, \, \ell \in \mathbb{Z} \right\}
\end{aligned}$$

and show that $F_{\tau,\gamma} \subseteq E_{\tau,\gamma}$ and that for each $\alpha \in E_{\tau,\gamma}$ there exists $\gamma_\alpha > 0$ with $\alpha \in F_{\tau,\gamma_\alpha}$ and with $\omega = (\alpha, 1) \in D_{\tau,\gamma_\alpha}$ and there exist $\gamma_i > 0$, $i = 1, 2$ such that $F_{\tau,\gamma} \subseteq G_{\tau,\gamma_1} \subseteq F_{\tau,\gamma_2}$. What is the relation between $F_{\tau,\gamma}$, $n = 1$ and the set

$$H_{\tau,\gamma} \quad = \quad \left\{ \alpha \in \mathbb{R} \ \middle| \ |\alpha - \frac{\ell}{k}| \geq \frac{\gamma}{k^{\tau+1}} \text{ for all } 0 \neq k \in \mathbb{Z}, \, \ell \in \mathbb{Z} \right\} \ ?$$

## 2 Rotating pendulum

Analyse the dynamics of the rotating pendulum $\ddot{x} = M - \sin x$ in dependence of $M \in \mathbb{R}$. Identify the values of the parameter $M$ where the dynamical behaviour of the system changes and give for each of the resulting open regions in parameter space at least one phase portrait. Is it possible to write the system as a Hamiltonian system on the cyinder $S^1 \times \mathbb{R}$ ? In that case write down the Hamiltonian function.

## 3 Circle mappings

Let $f : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ be an orientation-preserving homeomorphism of the circle with rotation number $\varrho(f) = 0$. Prove that $f$ has a fixed point.

## 4 Bound on derivative

Let $f$ be a $2\pi$-periodic analytic function defined and bounded in $\Pi_\rho = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \rho\}$. Prove that $\|d^m f/dz^m\|_{\rho-\delta} \leq C\|f\|_\rho \delta^{-m}$ for any $0 < \delta < \rho$, where $\|f\|_\rho = \sup\{|f(z)| \mid z \in \Pi_\rho\}$ is the supremum-norm of $f$ on the complex strip $\Pi_\rho$, $\rho > 0$.

## 5 Small divisors in analytic linearization

Let $F : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic mapping defined near $z = 0$ by $F(z) = \lambda z + f(z)$, where $0 \neq \lambda \in \mathbb{C}$, $f(0) = f'(0) = 0$. Consider the problem of local analytic conjugacy of $F$ to its linear part $A : \mathbb{C} \longrightarrow \mathbb{C}$, $A(z) = \lambda z$, i.e. the existence of an analytic diffeomorphism $H : \mathbb{C} \longrightarrow \mathbb{C}$, $H(z) = z + h(z)$ defined near $z = 0$ and such that $h(0) = h'(0) = 0$ and

$$F \circ H \quad = \quad H \circ A$$

in a neighbourhood of $z = 0$.

1. Derive the corresponding linear homological equation for $h$.
2. Show that its solution via Taylor series leads to a small divisor problem. How does this problem differ from that for the circle mappings ?
3. Explain, which $\lambda$'s should be called "resonant" and what would it mean for $\lambda = \mathrm{e}^{2\pi \mathrm{i} \alpha}$ to be "non-resonant".

## 6 Equivalent norms

Let $A$ be a real $m \times m$ matrix with eigenvalues $z_j \in \mathbb{C},\ j = 1, 2, \ldots, m$. Define its *spectral radius* $\rho := \max\{|z_1|, |z_2|, \ldots, |z_m|\}$.

1. Prove that for any $\lambda > \rho$ there exists $C \geq 1$ such that for all $n \geq 0$ in any norm on $\mathbb{R}^m$ and all $x \in \mathbb{R}^m$ the inequality

   $$\|A^n x\| \quad \leq \quad C\lambda^n \|x\|$$

   holds. *Hint*: You can use *Gelfand's Formula*:

   $$\rho \quad = \quad \lim_{n\to\infty} \|A^n\|^{1/n} \quad = \quad \inf_{n\geq 1} \|A^n\|^{1/n} \quad ,$$

   where $\| \cdot \|$ is the corresponding operator norm.

2. Prove that for any $\lambda$ with $\rho < \lambda < 1$ there exists a norm on $\mathbb{R}^m$ such that for all $n \geq 0$ and all $x \in \mathbb{R}^m$ the inequality

$$\|A^n x\| \quad \leq \quad \lambda^n \|x\|$$

holds.

3. Suppose that $A$ is invertible and $|\mu_j| \neq 1$ for $j = 1, 2, \ldots, m$. Prove that there exists a norm on $\mathbb{R}^m$ and $0 < \lambda < 1$ such that for all $n \geq 0$ one has

$$\begin{aligned} \|A^n x\| \quad &\leq \quad \lambda^n \|x\| \text{ for all } \quad x \in E^s \\ \|A^{-n} x\| \quad &\leq \quad \lambda^n \|x\| \text{ for all } \quad x \in E^u \ , \end{aligned}$$

where $E^s$ and $E^u$ are the generalized stable and unstable eigenspaces of $A$, respectively.

4. Give a definition for a hyperbolic invariant set $\Lambda$ of a diffeomorphism $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ using the notion of equivalent norms.

# 7 Eigenvalues of the commutator

Let $A$ be an $m \times m$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Consider the linear mapping on the space of all $m \times m$ complex matrices defined by the formula

$$X \quad \mapsto \quad \mathrm{ad}_A X \ = \ AX - XA \ .$$

Prove that all eigenvalues of $\mathrm{ad}_A$ are given by the differences

$$\lambda_j \ - \ \lambda_k \ ,$$

where $j, k \in \{1, 2, \ldots, m\}$.

# 8 Miniversal deformation of a Jordan block

Prove that the deformation

$$B(\mu) \quad = \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} \ , \quad \mu \in \mathbb{C}^4$$

is equivalent to a deformation induced from the deformation

$$A(\lambda) \quad = \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda_1 & \lambda_2 \end{pmatrix} \ , \quad \lambda \in \mathbb{C}^2$$

by explicitly constructing a smooth deformation $C(\mu)$ of the unit matrix and a smooth mapping $\phi : \mathbb{C}^4 \longrightarrow \mathbb{C}^2$, $\mu \mapsto \lambda = \phi(\mu)$, such that

$$B(\mu) \quad = \quad C(\mu)\, A(\phi(\mu))\, C^{-1}(\mu)$$

in a neighbourhood of $\mu = 0$.

## 9 Versal deformation for a codim 3 Hopf bifurcation

Consider the following three-parameter planar ODE written in the complex form

$$\dot{z} \;=\; (\beta_1 + \mathrm{i})z \;+\; \beta_2 z|z|^2 \;+\; \beta_3 z|z|^4 \;-\; z|z|^6 \;,$$

where $z = x + \mathrm{i}y \in \mathbb{C}$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$. Sketch its local bifurcation set near $\beta = 0$ together with the corresponding phase portraits in the $(x, y)$–plane:

1. Let $z = re^{\mathrm{i}\varphi}$ and $\rho = \frac{1}{2}r^2$. Write an equivalent system of differential equations using $(\rho, \varphi)$ as new coordinates and observe that it consists of two independent ODEs, for $\rho$ and $\varphi$.
2. Prove that $\rho = 0$ gives the only equilibrium of the original planar system. Establish a correspondence between the $\rho$–values for which $\dot{\rho} = 0$ and periodic orbits of the planar system.
3. Which equilibrium bifurcation occurs in the system, when one crosses the parameter plane $\mathcal{N}$ defined by $\beta_1 = 0$ ? How does the nature of this bifurcation depend on the sign of $\beta_2$ ?
4. Prove that there exists a bifurcation surface $\mathcal{K}$ in the $\beta$–space at which a *periodic saddle-node* bifurcation occurs, i.e. two periodic orbits of the system collide and disappear. Sketch an intersection of $\mathcal{K}$ with the plane $\beta_3 = \varepsilon$ near $\beta_1 = \beta_2 = 0$ for small fixed $\varepsilon < 0$ and $\varepsilon > 0$. *Hint*: You may first neglect the $\rho^3$–term.
5. Characterise a curve $\mathcal{C}$ in the $\beta$–space on which three periodic orbits of the system simultaneously collide, i.e. the system has a *triple* periodic orbit. Argue that along this curve the surface $\mathcal{K}$ forms a sharp edge, so that the intersection of $\mathcal{K}$ with the plane $\beta_3 = \varepsilon$ has a *cusp* point for small fixed $\varepsilon > 0$.
6. Sketch the intersection of the bifurcation set with the plane $\beta_3 = \varepsilon$ for $\varepsilon = 0$, as well as for small fixed $\varepsilon < 0$ and $\varepsilon > 0$. Draw the corresponding phase portraits in all parameter domains.
7. Visualize the obtained bifurcation set in the 3D parameter space.

## 10 Floquet exponents

For $T \in GL_n(\mathbb{R})$ show that $T$ has a real logarithm, i.e. $T = \exp A$ with $A \in M_{n\times n}(\mathbb{R})$, if and only if there exists $S \in GL_n(\mathbb{R})$ with $S^2 = T$. Use this to relate the eigenvalues of $T$, the Floquet multipliers, to the eigenvalues of $A$, the Floquet exponents. *Hint:* First note that also $T \in GL_n(\mathbb{C})$ and compute the complex logarithm. Assume (if necessary) that $T$ is semi-simple.

## 11 Ramified torus bundle

The aim of this exercise is to explore the phase space structure around an equilibrium of a Hamiltonian system in $n$ degrees of freedom. Sometimes you will need to make extra assumptions on the coefficients of $H$ expanded in certain co-ordinates: instead of discussing the possible cases always make the strongest assumptions. (See part $(i)$ for an example.)

$(i)$ Let the origin $(0,0) \in \mathbb{R}^{2n} = T^*\mathbb{R}^n$ be a minimum of the Hamiltonian function $H$. Show that there is a (linear) change to canonical co-ordinates $(x,y)$ such that $H = H_0^0 + \mathcal{O}(3)$ where

$$H_0^0(x,y) = \sum_{j=1}^{n} \alpha_j \frac{x_j^2 + y_j^2}{2}$$

with all $\alpha_j > 0$. (Next to the assumption that the equilibrium $(0,0)$ is a minimum with $H(0,0) = 0$ here the stronger assumption that $D^2H(0,0)$ is positive definite is necessary.)

$(ii)$ Show that the linear Hamiltonian system $X_{H_0^0}$ has $n$ one-parameter families of elliptic periodic orbits, an $n$–parameter family of invariant $n$–tori and $\binom{n}{k}$ $k$–parameter families of elliptic invariant $k$–tori, $k = 2, \ldots, n-1$.

Assuming that, given $m \in \mathbb{N}$, there are no (low order) resonances

$$h_1\alpha_1 + \ldots + h_n\alpha_n = 0 \;, \quad h \in \mathbb{Z}^n \;, \; |h| = |h_1| + \ldots + |h_n| \leq 2m+3$$

among the (normal) frequencies, a normalization procedure (that you do *not* have to carry out) yields a non-linear change to canonical co-ordinates $(q,p)$ such that

$$H = \sum_{\ell=0}^{m} \frac{H_0^{2\ell}}{\ell!} + \mathcal{O}(2m+4)$$

with

$$H_0^{2\ell}(q,p) = \sum_{i_1+\ldots+i_n=\ell+1} A_{i_1,\ldots,i_n} \prod_{j=1}^{n} \left( \frac{p_j^2+q_j^2}{2} \right)^{i_j} .$$

$(iii)$ Show that the Hamiltonian system $X_{\mathcal{H}}$ defined by the normal form truncation $\mathcal{H} = H_0^0 + H_0^2$ also has the families of periodic orbits and invariant tori of $X_{H_0^0}$ found in $(ii)$.

$(iv)$ Show persistence of all elliptic periodic orbits from the normal form truncation $\mathcal{H}$ to the full Hamiltonian $H$ in a sufficiently small neighbourhood of the origin.

$(v)$ Use action angle co-ordinates $(\varphi_j, \tau_j)$ with $\tau_j = \frac{p_j^2+q_j^2}{2}$ to write down the flow on the invariant $n$–tori of $X_{\mathcal{H}}$.

($vi$) Show persistence of 'many' invariant $n$–tori from the normal form truncation $\mathcal{H}$ to the full Hamiltonian $H$ in a sufficiently small neighbourhood of the origin.

($vii$) Show that the Hamiltonian system $X_{\mathcal{H}}$ defined by the normal form truncation

$$\mathcal{H} \quad = \quad \sum_{\ell=0}^{m} \frac{H_0^{2\ell}}{\ell!}$$

also has the families of periodic orbits and invariant tori of $X_{H_0^0}$ found in ($ii$).

($viii$) Does using the normal form truncation from ($vii$) yield better results in ($iv$) and ($vi$) ?

($ix$) Show persistence of 'many' elliptic invariant $k$–tori in a $k$–parameter family from the normal form truncation $\mathcal{H}$ to the full Hamiltonian $H$ in a sufficiently small neighbourhood of the origin.

# 12 Swallowtail

Determine the bifurcation diagram of the 3–parameter family

$$H_\lambda(x, y) \quad = \quad \frac{1}{2}y^2 \; + \; \frac{1}{5!}x^5 \; + \; \lambda_1 x \; + \; \frac{\lambda_2}{2}x^2 \; + \; \frac{\lambda_3}{6}x^3$$

of Hamiltonian systems in one degree of freedom.

# 13 Main tongue of Arnold family

For the Arnold family

$$\begin{array}{rccl} P_{\alpha,\varepsilon} \; : & \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \\ & x & \mapsto & x + 2\pi\alpha + \varepsilon \sin x \end{array}$$

of circle maps consider the region in the $(\alpha, \varepsilon)$–plane where $P_{\alpha,\varepsilon}$ has a fixed point. Compute the boundary of this region, describe (also sketch) the dynamics on both sides of and on a boundary curve and explain which bifurcation takes place.

# 14 Degenerate pitchfork bifurcation

To study the degenerate period-doubling and frequency-halving bifurcations consider a 1–dimensional $\mathbb{Z}_2$–symmetric vector field with an equilibrium at the origin (for which the linearization is zero) that has a vanishing first Lyapunov coefficient (this is the coefficient that distinguishes between the supercritical and the subcritical case of the bifurcation, $\beta_2$ in exercise 9). The aim is to formulate genericity conditions under which a 1–parameter unfolding is locally topologically equivalent to a standard family.

1. Use the co-ordinate $z$ in which the $\mathbb{Z}_2$–symmetry is generated by $z \mapsto -z$ (whence the vector field takes a particularly simple form) to expand the right hand side.

2. Truncate the expansion to lowest significant order and identify the two coefficients that should serve as parameters $\lambda_1, \lambda_2$.

3. Scale away remaining coefficients where possible. Make your own choice of the second Lyapunov coefficient ($\beta_3$ in exercise 9) to continue with.

4. Give the bifurcation diagram of the resulting standard family $\dot{z} = h_\lambda(z)$.

5. Show that the initial unfolding is locally topologically equivalent to the dynamics of your standard family for parameter values in the open domains of the bifurcation diagram.

6. Show that the initial unfolding is locally topologically equivalent to your standard family for 1–parameter subfamilies crossing the curves of the bifurcation diagram.

7. Show that the initial unfolding is locally topologically equivalent to your standard family.

In case you encounter problems along the way that you cannot solve, try to describe these problems as accurately as possible and make assumptions (educated guesses) if necessary to continue.

8. Same as above, but this time for the 'periodically driven' vector field

$$\begin{aligned} \dot{x} &= \omega + \mathcal{O}(z) \\ \dot{z} &= h_\lambda(x, z) \ . \end{aligned}$$

   Explain what the results mean for the degenerate period-doubling bifurcation. *Hint:* use normal form theory in 1. to derive $x$–independence of the lower order terms.

9. How far can you get with a quasi-periodic version (and thus with the frequency-halving bifurcation) ? Where do the problems occur?