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Seminar Bifurcation Analysis

Simplest Normal form for the Brusselator

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As a concrete example we study the Brusselator, which is a toy model that models an autocatalytic reaction. This type of a chemical reaction is called autocatalytic, as the chemical reaction produces the catalyst as a byproduct, hence increasing the reaction rate. This is able to produce complex dynamics, which includes oscillating behaviour. First observed by Boris Belousov in 1951, his work on oscillating reactions remained unpublished as the scientific community deemed oscillations in chemical reactions to be prohibited by the laws of thermodynamics. However, in 1961 Zhabotinksy reproduced Belousov's work hence proving that they were indeed possible. Now these models are well studied from both a mathematical and a chemical point of view. To see an example of this type of reaction, we refer to [Ken11]. The Brusselator is a toy model derived from the BZ-reaction. The chemical reaction is governed by the differential equation

$$\begin{aligned}\dot{x} &= A + x^2y - Bx - x\\ \dot{y} &= Bx - x^2y.\end{aligned}$$

Where $A > 0$ is a fixed constant and B is a bifurcation parameter.

Exercise 1 Compute the equilibrium and the parameter value $B_0 = B_0(A)$ at which the system exhibits the Andronov Hopf bifurcation. Lastly, explain why diagonalizing the linear part of the vector field makes the process of normalisation easier.

Translating the equilibrium to the origin and diagonalizing the linear part of the vector field we obtain the following vector field.

$$\begin{aligned}\dot{x} &= x\mathrm{i} + x^3(-\tfrac{1}{4} - \tfrac{1}{4}\mathrm{i}) + x^2y(-\tfrac{3}{4} + \tfrac{1}{4}\mathrm{i}) + x^2(\tfrac{1}{2} - \tfrac{1}{2}\mathrm{i})\\ &\quad + xy^2(-\tfrac{1}{4} + \tfrac{3}{4}\mathrm{i}) + y^3(\tfrac{1}{4} + \tfrac{1}{4}\mathrm{i}) + y^2(\tfrac{1}{2} - \tfrac{1}{2}\mathrm{i})\\ \dot{y} &= -y\mathrm{i} + x^3\left(\tfrac{1}{4} - \tfrac{1}{4}\mathrm{i}\right) + x^2y\left(-\tfrac{1}{4} - \tfrac{3}{4}\mathrm{i}\right) + x^2\left(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}\right)\\ &\quad + xy^2\left(-\tfrac{3}{4} - \tfrac{1}{4}\mathrm{i}\right) + y^3\left(-\tfrac{1}{4} + \tfrac{1}{4}\mathrm{i}\right) + y^2\left(\tfrac{1}{2} + \tfrac{1}{2}\mathrm{i}\right)\end{aligned}$$

Exercise 2 Set up the homological equation for a general quadratic transformation and solve it. Then construct the transformation that will remove the quadratic terms. Note that you do not have to apply the transformation to the system.

After applying this transformation to the system we obtain the following vector field, normalised up to the quadratic terms,

$$\dot{x} = +\mathrm{i}x + x^3\left(-\frac{5}{4} - \frac{1}{4}\mathrm{i}\right) + x^2\,y\left(-\frac{3}{4} - \frac{1}{12}\mathrm{i}\right) + x\,y^2\left(\frac{1}{12} + \frac{3}{4}\mathrm{i}\right) + y^3\left(\frac{1}{4} + \frac{5}{4}\mathrm{i}\right)$$
$$\dot{y} = -\mathrm{i}y + x^3\left(\frac{1}{4} - \frac{5}{4}\mathrm{i}\right) + x^2\,y\left(\frac{1}{12} - \frac{3}{4}\mathrm{i}\right) + xy^2\left(-\frac{3}{4} + \frac{1}{12}\mathrm{i}\right) + y^3\left(-\frac{5}{4} + \frac{1}{4}\mathrm{i}\right)$$

Exercise 3 Repeat the process with a general cubic transformation and compute the classical normal form up to third order. Next, convert the system to polar coordinates and decide on the stability of the Hopf bifurcation at the critical value.

Exercise 4 Illustrate your analysis for $A = 1$ by construction the phase portrait of the Brusselator at $B = B_0$ and for $B < B_0$ and $B > B_0$ with small $\|B - B_0\|$.

References

[Ken11] Tim Kench. *The Belousov-Zhabotinsky Oscillating Reaction*. 2011. URL: https://www.youtube.com/watch?v=PpyKSRo8Iec (visited on).