Seminar Bifurcation
Analysis

Ernst Röell
4149661
Fall 2018

Homework 4

Due Date: 22 October 2018
email: `ernstroell@gmail.com`

Exercise 1 This exercise explores the theory in the setting of the matrix Lie group $GL(2,\mathbb{R})$, the set of 2×2 invertible matrices. You may assume without proof that any one parameter subgroup in $GL(2,\mathbb{R})$ is the exponential of a matrix. That is, let $\varphi^t : \mathbb{R} \to GL(2,\mathbb{R})$ be the one parameter subgroup, then there exists a matrix $A \in gl(2,\mathbb{R})$ s.t. $\varphi^t = e^{tA}$. Note that if we let this subgroup act on $\mathbb{R}^2$, we obtain a flow. Recall from the lecture that for a Lie group G the conjugation by $g \in G$ operator was defined as

$$\mathsf{C}_g : G \to G, \quad x \mapsto g^{-1}xg.$$

In the case of $GL(2,\mathbb{R})$ the conjugation operator is given by

$$\mathsf{C}_P X = P^{-1}XP.$$

Also recall that the Lie algebra of $GL(2,\mathbb{R})$ is given by $gl(2,\mathbb{R})$, the set of all 2×2 matrices.

(a) Show that $gl(2,\mathbb{R})$ with the standard commutator bracket $[A, B] = AB - BA$ is a Lie algebra.

(b) Show that the similarity by $P \in GL(2,\mathbb{R})$ operator, acting on the Lie algebra $gl(2,\mathbb{R})$ is given by

$$\mathsf{C}_P : gl(2,\mathbb{R}) \to gl(2,\mathbb{R}), X \mapsto P^{-1}XP.$$

Do this, by considering a general one parameter subgroup φ^t in $GL(2,\mathbb{R})$. Then apply conjugation by a matrix $P \in GL(2,\mathbb{R})$ and differentiate this relation at $t = 0$.

(c) Lastly, show that the Lie bracket is actually the standard commutator bracket. Do this in the following way. For a one parameter subgroup ψ^t and an element $A \in gl(2,\mathbb{R})$, apply a similarity by ψ^t to A and then differentiate this relation at $t = 0$.

Exercise 2 Consider the flow of the planar linear system given by

$$\varphi^t(x_1, x_2) = \begin{pmatrix} e^{-t} & e^t - e^{-t} \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(a) Show that there exists a matrix $P \in GL(2,\mathbb{R})$ so that there is a linear change of coordinates to the flow

$$\psi^t(y_1, y_2) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

For both systems, make a phase portrait.

(b) As discussed in lecture, each flow arises as the solution of a vector field. For both systems above, show that the corresponding differential equations are

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

(c) Show that there exists a $P \in GL(2,\mathbb{R})$ that transforms one differential equation into the other. That is, there is a linear change of coordinates $y = Px$ that transforms the one system into the other. Hint: For both obtained systems, compute the eigenvalues.