

### Homework 5

(i) Show that the space of axially symmetric vector fields is generated by the three vector fields

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} , \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial z} ,$$

meaning that the most general  $\mathbb{S}^1$ -equivariant vector field has the form

$$f(\tau, z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + g(\tau, z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + h(\tau, z)\frac{\partial}{\partial z} \quad (1)$$

where  $\tau = \frac{1}{2}(x^2 + y^2)$ .

(ii) Show that for (1) to be volume-preserving, the coefficient functions have to satisfy

$$2\frac{\partial \tau g}{\partial \tau} + \frac{\partial h}{\partial z} = 0 . \quad (2)$$

(iii) In the invariants  $\tau$  and  $z$  the vector field (1) reduces to

$$\dot{\tau} = 2\tau g(\tau, z) \quad (3a)$$

$$\dot{z} = h(\tau, z) \quad (3b)$$

whence the line  $\{\tau = 0\}$  is always invariant — as expected from the  $\mathbb{S}^1$ -symmetry. Use (2) to show that the equations of motion (3) are Hamiltonian with Hamiltonian function

$$H(\tau, z) = \int_0^z 2\tau g(\tau, \tilde{z}) d\tilde{z} - \int_0^\tau h(\tilde{\tau}, 0) d\tilde{\tau} .$$