Normal form of the Hopf singularity

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Definition (Continuous-time dynamical system)

A continuous-time dynamical system consists of a one-parameter family of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$, where $t \in \mathbb{R}$ that form a one-parameter group. That is, $\varphi_s \circ \varphi_t = \varphi_{s+t}$ and $\varphi_0 = Id_{\mathbb{R}^n}$. We call the map φ_t the flow.

Theorem (Hartman-Grobman Theorem)

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be continuously differentiable, and $x_0 \in U$ be a hyperbolic fixed point of f. Then there exists neigbourhoods U_1, U_2, V_1, V_2 of x_0 and a homeomorphism $h : U_1 \cup U_2 \to V_1 \cup V_2$ such that the following diagram commutes.

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ \downarrow h & & \downarrow h \\ V_1 & \xrightarrow{Df_{x_0}} & V_2 \end{array}$$

See [KH95] on page 261.

Hopf Bifurcation

Consider the smooth system

$$\dot{x_1} = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x_2} = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)$$

or in polar coordinates

$$\dot{\rho} = \rho(\alpha - \rho^2)$$
$$\dot{\varphi} = 1.$$

Then $\alpha = 0$ is a critical value.

Lemma ([Kuz04], Lemma 3.2)

The system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + O(\|x\|^4)$$

is locally topologically equivalent to the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 . \end{pmatrix}$$

Proof I

We only prove the existence and uniqueness of the cycle. We follow [Kuz04] and the rest of the proof can be found therein. First write the system in polar coordinates as

$$\dot{\rho} = \rho(\alpha - \rho^2) + \Phi(\rho, \varphi)$$
$$\dot{\varphi} = 1 + \Psi(\rho, \varphi).$$

Now reparametrize time so that the time it takes for an orbit starting at $\varphi = 0$ to return is set to 1. Then any orbit starting at $(\rho, \varphi) = (\rho_0, 0)$ has ρ satisfying

$$\frac{d\rho}{d\varphi} = \frac{\rho(\alpha - \rho^2) + \Phi(\rho, \varphi)}{1 + \Psi(\rho, \varphi)} + R(\rho, \varphi).$$

Because $\rho(\varphi, 0) = 0$ we can write the Taylor expansion of $\rho(\varphi, \rho_0)$ as

$$\rho = u_1(\varphi)\rho_0 + u_2(\varphi)\rho_0^2 + u_3(\varphi)\rho_0^3 + \dots .$$

Proof II

Now substitute this into the above and solve the linear differential equation at each order or ρ_0 with initial condition $u_1(0) = 1, u_2(0) = 0, u_3(0) = 0$. The result is

$$u_1(\varphi) = e^{\alpha\varphi}, \quad u_2(\varphi) = 0, \quad u_3(\varphi) = e^{\alpha\varphi}\frac{1 - e^{2\alpha\varphi}}{2\alpha}.$$

Since these expressions are independent of $R(\rho, \varphi)$, the Poincaré return map has the form

$$\rho_1 = e^{2\pi\alpha}\rho_0 - e^{2\pi\alpha}\left[2\pi + O(\alpha)\right]\rho_0^3 + O(\rho_0^4).$$

Now this map can be easily analyzed for sufficiently small ρ_0 and α. For $\alpha < 0$ it only has a trivial fixed point and for small $\alpha > 0$ there is another fixed point, indicating the existence of a cycle.

Definition (Smooth equivalence)

Let $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ be two flows. Then we call φ_t and ψ_t smoothly equivalent if there exists a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{h} & \mathbb{R}^n \\ \Big\downarrow{\scriptstyle \varphi_t} & & \Big\downarrow{\scriptstyle \psi_t} \\ \mathbb{R}^n & \xrightarrow{h} & \mathbb{R}^n \end{array}$$

See Definition 2.2.1, [KH95] on page 64.

Remark

- *Note that smooth equivalence is a equivalence relation and the goal is to find the simplest representative.*
- *Near identity transformations are local diffeomorphisms by the inverse function theorem.*

Classical Normal Form

Any smooth system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \dots$$

can be formally transformed to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{k=1}^{\infty} (x_1^2 + x_2^2)^k \left[C_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_k \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right],$$

or in polar coordinates

$$\begin{pmatrix} \dot{\rho} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=1}^{\infty} \begin{pmatrix} C_k \rho^{2k+1} \\ D_k \rho^{2k} \end{pmatrix}, C_k, D_k \in \mathbb{R}.$$

Theorem

Denote by V_k the span of the set of homogeneous polynomials of degree $k+1$. The coordinate change $x = y + h_k(y)$ will take the system

$$\dot{x} = v_0(x) + v_1(x) + v_2(x) + \dots, \quad v_k(x) \in V_k$$

into the system

$$\dot{y} = v_0(y) + v_1(y) + v_2(y) + \cdots + v_{k-1}(y) + b_k(y) + \cdots,$$

where

$$b_k(y) = v_k(y) + Dv_0(y)h_k(y) - Dh_k(y)v_k(y).$$

Definition

We define an operator, called the homological operator, as the map

$$L_{v_0}^k : V_k \to V_k, \quad h_k \mapsto Dv_0(x)h_k(x) - Dh_k(x)v_0(x),$$

where V_k is the vector space spanned by the homogeneous polynomials of degree $k+1$.

Proof I

We follow the analysis in [Kuz04] on page 108. By the Inverse Function Theorem the function $x = y + h_k(y)$ is invertible on a small neigbourhood of the origin. Now differentiating the change of coordinates with respect to time yields

$$\dot{x} = \dot{y} + Dh_k(y)\dot{y}.$$

The inverse can be written and expanded as

$$\dot{y} = (I - Dh_k(y) + O(\|y\|^{k+1}))\dot{x},$$

which is also smooth. Here I denotes the identity on $\mathbb{R}^n$.

Proof II

Plugging the definitions in and rewriting then yields

$$
\begin{aligned}
\dot{y} &= (I - Dh_k(y) + O(\|y\|^{k+1}))(v_0(y + h_k(y)) + v_1(y + h_k(y)) + \cdots) \\
&= (I - Dh_k(y) + O(\|y\|^{k+1}))(v_0(y) + v_0(h_k(y)) + \sum_{i=1}^{k} v_i(y) + \ldots) \\
&= v_0(y) + \sum_{i=1}^{k-1} v_i(y) + v_k(y) + v_0(h_k(y)) - Dh_k(y)v_0(y) + O(\|y\|^{k+1}).
\end{aligned}
$$

This proves the result.

Theorem (Poincaré 1879)

There is a polynomial change of coordinates

$$x = y + h_1(y) + h_2(y) + \cdots + h_m(y), \quad h_k \in V_k$$

that transforms the smooth system

$$\dot{x} = v_0(x) + v_1(x) + v_2(x) + \ldots, \quad v_k(x) \in V_k$$

into

$$\dot{y} = v_0(y) + r_1(y) + r_2(y) + \cdots + r_m(y) + O(\|y\|^{m+1}), \quad r_k \in V_k.$$

where each r_k *contains only resonant terms of degree* $k+1$*. The resonant terms are the terms that lie in the complement of the image of the homological operator.*

Example

Consider a general system governed by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + v_1(x) + \dots, \quad v_k \in V_k.$$

First apply a near identity transformation

$$x = y + h_1(y), \quad h_1 \in V_1,$$

or explicitly written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} c_{120}x_1^2 + c_{111}x_1x_2 + c_{102}x_2^2 \\ c_{220}x_1^2 + c_{211}x_1x_2 + c_{202}x_2^2 \end{pmatrix}.$$

The first number in the index denotes either the first or second component, the second number the exponent in x_1 and the third number the exponent in x_2.

Apply the general quadratic transformation and gather all quadratic terms. This yields

$$\begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{120} \\ c_{111} \\ c_{102} \\ c_{220} \\ c_{211} \\ c_{202} \end{pmatrix} = \begin{pmatrix} a_{120} - b_{120} \\ a_{111} - b_{111} \\ a_{102} - b_{102} \\ a_{220} - b_{220} \\ a_{211} - b_{211} \\ a_{202} - b_{202} \end{pmatrix}.$$

- The terms c_{***} are the terms we transform with,
- a_{***} are the original terms,
- b_{***} are the remaining terms.

As the matrix is invertible, the quadratic terms can be removed.

Now apply a third order transformation

$$x = y + h_2(y), \quad h_2 \in V_2.$$

The equation now becomes

$$\begin{pmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{103} \\ c_{112} \\ c_{121} \\ c_{130} \\ c_{203} \\ c_{212} \\ c_{221} \\ c_{230} \end{pmatrix} = \begin{pmatrix} \tilde{a}_{103} - b_{103} \\ \tilde{a}_{112} - b_{112} \\ \tilde{a}_{121} - b_{121} \\ \tilde{a}_{130} - b_{130} \\ \tilde{a}_{203} - b_{203} \\ \tilde{a}_{212} - b_{212} \\ \tilde{a}_{221} - b_{221} \\ \tilde{a}_{230} - b_{230} \end{pmatrix}.$$

This matrix is not invertible, so some cubic terms will remain.

The kernel of this operator is spanned by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

As the choice of basis is not unique, different choices will result in different but equivalent normal forms.

Thus, the normal form up to third order is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \left[C_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_1 \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right] + O(\|x\|^4).$$

The real constants C_1 and D_1 are in terms of the original coefficients.

Definition (Semisimple Operator)

Let V be a finite dimensional vector space and let $A \in \operatorname{End} V$. Then A is said to be semisimple if the matrix associated to the operator is diagonalisable over the complex numbers.

Lemma (Semisimple splitting)

Let V be a real finite dimensional vector space. If the operator $L : V \to V$ is semi-simple, then V can be decomposed into:

$$V = \operatorname{im} L \oplus \ker L$$

Proof.

First choose a basis in which L is diagonal. Then the image of L consists of the span of the nonzero eigenvalues. The kernel now consists of those eigenvectors who's corresponding eigenvalue is zero. The Lemma is now immeadiate. $\square$

Lemma (Splitting of L)

Let $A \in \operatorname{End} V$ *be semi-simple, then the homological operator*

$$L_A : V_k \to V_k, \quad h_k(x) \mapsto Ah_k(x) - Dh_k(x)Ax$$

is semi-simple for every $k \in \mathbb{N}$*. Moreover, if the* A *is diagonal, then the homological operator is diagonal as well.*

Remark

A monomial $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ *is denoted by* x^m*, where* $x \in \mathbb{R}^n$ *and* $m \in \mathbb{N}_0^n$*.*

For more details, see [Mur06].

Proof I

The strategy is to compute the eigenvalues, thereby showing diagonalizability and hence semi-simplicity. Let $A = T^{-1}\Lambda T$ be the diagonalization and consider $x^m e_i$. Now we consider the transformation $x = Ty$. When applying such a transformation, notice that for a general vector field $\dot{x} = a(x)$, the transformation yields $\dot{y} = T^{-1}a(Ty) = b(y)$. Plugging this in yields

$$\begin{aligned}
T^{-1}L_A a(Ty) &= T^{-1}Aa(Ty) - T^{-1}a'(Ty)ATy \\
&= T^{-1}ATT^{-1}a(Ty) - T^{-1}a'(Ty)ATy \\
&= \Lambda T^{-1}a(Ty) - T^{-1}a'(Ty)T\Lambda y \\
&= \Lambda b(y) - (T^{-1}a(Ty))'\Lambda y \\
&= \Lambda b(y) - b'(y)\Lambda y \\
&= L_\Lambda b(y)
\end{aligned}$$

Proof II

This shows the second claim. Now choose $b(y) = y^m e^i$. Then it follows that

$$\begin{aligned} L_\Lambda y^m e_i &= \Lambda y^m e_i - D y^m e_i \Lambda y \\ &= (\lambda_i - \langle m, \lambda \rangle), \end{aligned}$$

where λ denotes the vector with the eigenvalues as entries. Hence we found k eigenvalues of the transformed operator. Then by inverting the transformation and setting $a(y) = Tb(T^{-1}y)$, we find the eigenvectors of the original problem with the same eigenvalues. Hence we obtain k eigenvalue eigenvector pairs, which equals the dimension of V_n and it follows that the homological operator is diagonalisable.

In light of the previous lemma, the strategy is to first diagonalize the linear part of the vector field as this guarantees that the monomials are eigenvectors and then find the zero eigenvalues. The zero eigenvalues are found by solving

$$\langle m, \lambda \rangle - \lambda_1 = 0,$$
$$\langle m, \lambda \rangle - \lambda_2 = 0.$$

In the case of the Hopf singularity we set $\lambda_1 = i$ and $\lambda_2 = -i$. For the first equation, we then obtain

$$\begin{pmatrix} z_1^{m_2+1} z_2^{m_2} \\ 0 \end{pmatrix}.$$

The second yields the monomials

$$\begin{pmatrix} 0 \\ z_1^{m_1} z_2^{m_1+1} \end{pmatrix}.$$

It follows that the description of the classical normal form up to any degree is given by

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \sum_{k=1}^{\infty} (z_1 z_2)^k \left[C_k \begin{pmatrix} z_1 \\ 0 \end{pmatrix} + \overline{C_k} \begin{pmatrix} 0 \\ z_2 \end{pmatrix} \right]$$

where $C_k \in \mathbb{C}$. The transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

transforms the system back into

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{k=1}^{\infty} (x_1^2 + x_2^2)^k \left[\mathrm{Re}(C_k) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathrm{Im}(C_k) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right].$$

And this is exactly the result we got using the ad hoc methods, but to all orders.

Hypernormalisation

Consider the system in classical normal form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \sum_{k=0}^{\infty} (z_1 z_2)^k \left[C_k \begin{pmatrix} 0 \\ z_2 \end{pmatrix} + \overline{C_k} \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right]$$

and apply the transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + w_1 w_2 \left[c_{121} \begin{pmatrix} w_1 \\ 0 \end{pmatrix} + c_{212} \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \right] + \sum_{k=0}^{5} \begin{pmatrix} c_{1(5-k)(k)} w_1^{5-k} w_2^k \\ c_{2(5-k)(k)} w_1^{5-k} w_2^k \end{pmatrix}.$$

Note that the third order terms lie do not modify the third order terms, as they span the kernel of $L_{v_0}^2$.

The resulting equation is

$$\begin{pmatrix} 4ic_{150} \\ 2ic_{132} \\ 0 \\ -2ic_{241} \\ -4ic_{223} \\ -6ic_{205} \\ 6ic_{250} \\ 4ic_{232} \\ 2ic_{214} \\ 0 \\ -2ic_{123} \\ -4ic_{105} \end{pmatrix} = \begin{pmatrix} b_{150} \\ b_{141} \\ b_{132} + b_{121}\, c_{212} - b_{212}\, c_{121} \\ b_{123} \\ b_{114} \\ b_{105} \\ b_{250} \\ b_{241} \\ b_{232} \\ b_{223} - b_{121}\, c_{212} + b_{212}\, c_{121} \\ b_{214} \\ b_{205} \end{pmatrix}.$$

Hence the cubic terms have to be balanced so that

$$\begin{pmatrix} -b_{121} & b_{212} \\ b_{121} & -b_{212} \end{pmatrix} \begin{pmatrix} c_{212} \\ c_{121} \end{pmatrix} - \begin{pmatrix} b_{132} \\ b_{223} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that the coefficients $b_{***} \in \mathbb{C}$, while the coefficients $c_{***} \in \mathbb{R}$. Now it is evident that this equation has no solution. Hence either the real or imaginary part can be removed, and we choose the latter. In order to do so we first fill in the relations, which yields

$$\begin{pmatrix} -b_{121} & \overline{b_{121}} \\ b_{121} & -\overline{b_{121}} \end{pmatrix} \begin{pmatrix} c_{212} \\ c_{121} \end{pmatrix} = \begin{pmatrix} b_{132} \\ \overline{b_{132}} \end{pmatrix}.$$

Then by choosing the coefficients to be

$$c_{121} = -\frac{\mathrm{Im}(b_{132})}{2 * \mathrm{Im}(b_{121})} \quad c_{212} = -\frac{\mathrm{Im}(b_{132})}{2 * \mathrm{Im}(b_{121})}$$

we accomplish the task of removing the imaginary part of the equation.

The resulting normal form up to fifth order now becomes

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} b_{121} z_1^2 z_2 \\ b_{212} z_1 z_2^2 \end{pmatrix} + \begin{pmatrix} \mathrm{Re}(b_{132}) z_1^3 z_2^2 \\ \mathrm{Re}(b_{223}) z_1^2 z_2^2 \end{pmatrix} + \dots$$

Applying the same strategy, we use a transformation of the form

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + (w_1 w_2)^2 \left[c_{132} \begin{pmatrix} w_1 \\ 0 \end{pmatrix} + c_{223} \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \right] \\ + \sum_{k=0}^{7} \begin{pmatrix} c_{1(7-k)(k)} \xi^{7-k} \eta^k \\ c_{2(7-k)(k)} \xi^{7-k} \eta^k \end{pmatrix}. \end{aligned}$$

Applying this transformation and gathering the seventh order terms yields the following equation in matrix form.

$$\begin{pmatrix}
6i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6i
\end{pmatrix}
\begin{pmatrix} c_{170} \\ c_{161} \\ c_{152} \\ c_{143} \\ c_{134} \\ c_{125} \\ c_{116} \\ c_{107} \\ c_{270} \\ c_{261} \\ c_{252} \\ c_{243} \\ c_{234} \\ c_{225} \\ c_{216} \\ c_{207} \end{pmatrix}
=
\begin{pmatrix} \tilde{b}_{170} \\ \tilde{b}_{161} \\ \tilde{b}_{152} \\ \tilde{b}_{143} \\ \tilde{b}_{134} \\ \tilde{b}_{125} \\ \tilde{b}_{116} \\ \tilde{b}_{107} \\ \tilde{b}_{270} \\ \tilde{b}_{261} \\ \tilde{b}_{252} \\ \tilde{b}_{243} \\ \tilde{b}_{234} \\ \tilde{b}_{225} \\ \tilde{b}_{216} \\ \tilde{b}_{207} \end{pmatrix}.$$

It is noted though that the coefficients in the kernel of the seventh order terms on the right hand side are linear in the coefficients c_{132} and c_{223}, with matrix

$$\begin{pmatrix} -b_{121} - 2\,b_{212} & b_{121} \\ b_{212} & -2\,b_{121} - b_{212} \end{pmatrix}.$$

This matrix is nonsingular because we assumed that the third order terms had nonzero coefficients. Hence both coefficients c_{132} and c_{223} can be chosen so that the seventh order terms in the kernel of the seventh order homological operator can be completely removed.

Theorem

When $C_1 \neq 0$, the normal form up to all orders is given by:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} C_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \left[C_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_1 \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right] + C_2 (x_1^2 + x_2^2)^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

See [Yu99] for a more detailed treatment and proof of the theorem.

Theorem (Theorem 3, [YL02])

The simplest normal form of

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^2, \mu \in \mathbb{R}, f : \mathbb{R}^3 \to \mathbb{R}^2$$

given in polar coordinates is

$$\begin{aligned} \dot{\rho} &= \rho(\alpha\nu + C_1\rho^2) \\ \dot{\varphi} &= 1 + D_1\varphi^2 + D_2\varphi^4 + \dots \end{aligned}$$

up to any order, where $C_1 \neq 0$.

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