Lecture 1: Saddle homoclinic bifurcation

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1 Andronov-Leontovich thm (planar case)

In order to show the hyperbolic saddle bifurcation, I will use the Andronov-Leontovich theorem, which generally says the following:

Theorem 1.1. *(Andronov-Leontovich theorem, 2D)*
In a generic one-parameter family of vector fields in the plane, only 1 limit cycle can be created from a saddle loop

This theorem essentially describes the existence and stability of a periodic orbit as a hyperbolic equilibrium of a planar vector field undergoes a homoclinic bifurcation.
Now, what does this genericity means?

1.1 genericity

Definition 1.1. (Genericity assumptions 2D)
Suppose that vector field contained in a one-parameter family X_ε has homoclinic loop γ of hyperbolic saddle O. The corresponding parameter value, say $\varepsilon = 0$, will be called critical.
Suppose that the family X_ε satisfies the following *genericity assumptions*:

1. let $\lambda < 0 < \mu$ be the eigenvalues of the saddle O of the vector field X_0. Then λ/μ is irrational
2. in the family X_ε, the homoclinic loop occurs and breaks in a transversal manner as ε passes through zero. More precisely; let Γ be an oriented segented transversal to the field X_0 at a point of γ. Let $p_s(\varepsilon)$, $p_u(\varepsilon)$ be the first intersection points of the stable and unstable manifolds of O with Γ respectively. We denote the distance between $p_s(\varepsilon)$ and $p_u(\varepsilon)$ by $\rho(\varepsilon)$. The second genericity assumption means:
$$\frac{\mathrm{d}\rho(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \neq 0$$

1.2 bifurcations of homoclinic orbits of hyperbolic saddles

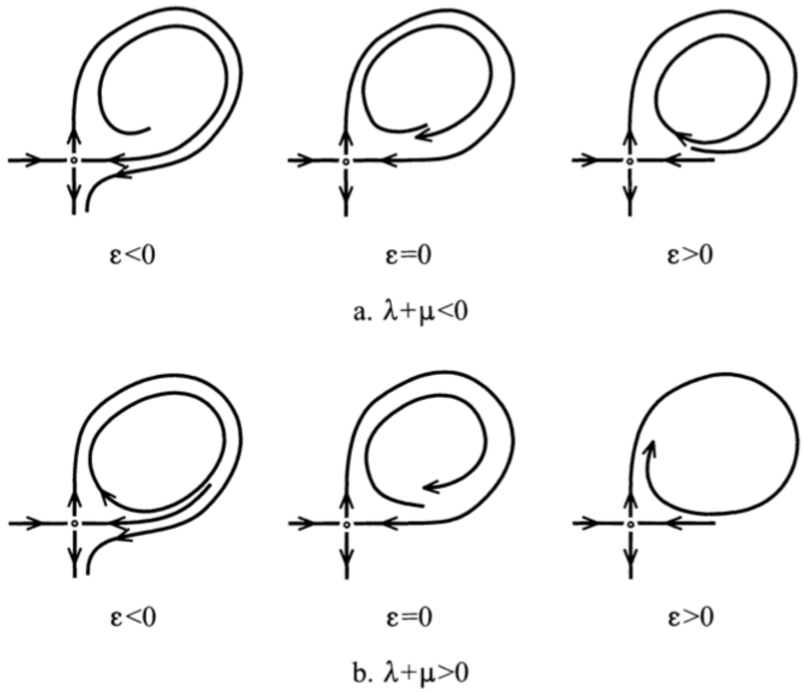
Now using the latter genericity assumptions we will rephrase the Andonov-Leontovich theorem for the planar case:

Theorem 1.2. *(rewrite A-L thm)*
Let the family X_ε satisfy the two genericty assumptions. and let σ be the saddle vale
Then $\exists$ U (neighborhood of γ) and E (neighborhood of $\varepsilon = 0$) satisfying the following conditions:

1. *$\sigma < 0 \Rightarrow \forall\varepsilon \in V$ lying on one side of 0, $\exists!$ stable periodic orbit in U converging to γ as ε approaches 0.*

   For all ϵ on the opposite side of zero, X_ε does not contain a periodic orbit
2. *If $\sigma > 0 \Rightarrow$ the same holds, but now unstable periodic orbit*

Now the saddle homoclinic bifurfaction looks as follows for this case:



1.3 Poincaré-map / correspondence map

Next, I will give an outline of the proof for the latter theorem and at the same time introduce normal forms and the correspondence map.
The goal is eventually to construct a poincare map near the orbit γ

Proof. This Poincare-map will be a composition of 2 functions; singular map (correspondence map) and a regular map.
First we take two segments transversal to the field X_0 at two points of γ that are close to the singular point O:

- Entrance segment T^+: intersecting the stable manifold of saddle at p
- Exit segment T^-: intersecting the unstable manifold of saddle at q

For planar hyperbolic saddle we have that a correspondence map is defined for any hyperbolic sector of the saddle.
The correspondence map: brings half-interval Γ^+ with vertex on stable manifold transversal to this manifold, to similar half interval Γ^- (vertex on unstable manifold)
Correspondence map is map along the phase curves of the field (lipshitz: map bounded away from 0 and ∞)
If vector field depends on parameter $\Rightarrow$ correspondence map depends on parameter:

$$\Delta_\varepsilon^{\text{sing}} : \Gamma^+ \to \Gamma^-$$

note: it can be shown that it is not well defined in the full neighborhood of zero in the product of the parameter and coordinate spaces, and its derivatives may tend to infinity at some points of the boundary of its domain. Furthermore, contraction map, tending to 0 as domain tends to p.
Regular map: diffeomorphism depending smoothly on ε. ($\Delta_\varepsilon^{\text{reg}}$)
- solutions of ode's are differentiable wrt initial conditions and parameters for sufficiently small $|\varepsilon|$:
Hence the positive half orbit starting at point of T^- near q, will intersect T^+ at some point near p.
Point on T^+ is image of initial point.

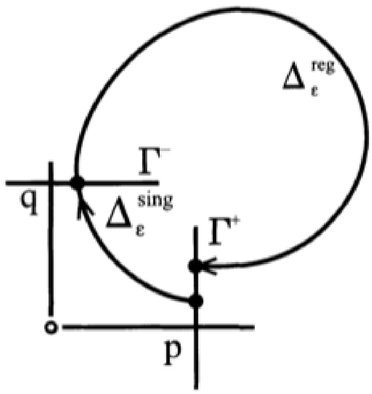


FIGURE 3.3. Definitions of regular and singular maps.

Poincaré map: $\Delta_\varepsilon = \Delta_\varepsilon^{\text{reg}} \circ \Delta_\varepsilon^{\text{sing}} \Rightarrow$ study fixed point of Δ_ε
Since singular map contraction $\Rightarrow$ Poincaré map is a contraction for small $\varepsilon \Rightarrow$ unique stable fixed point (if domain mapped to itself)
textbfStudy of correspondence map: Normal forms for perturbation of planar hyperbolic saddles and correspondence maps
Choose 'right' coordinates near O. There exists a finitely smooth family of charts in some neighborhood of O (call this D), transforming the initial system to linear system:

$$\begin{cases} \dot{x} = \lambda(\varepsilon)x \\ \dot{y} = \mu(\varepsilon)y \end{cases} \tag{1}$$

with: $\lambda(0) = \lambda < 0 < \mu = \mu(0)$ and assume normalization: $\{|x|, |y| \leq 1\} \subset D$
Choose entrance, exit segment and Γ^+ as follows:

$$T^+ = \{(x,y) \in D | x = 1\}$$
$$T^- = \{(x,y) \in D | y = 1\}$$
$$\Gamma^+ = \{(x,y) \in D | x = 1, y > 0\}$$

we have Equation (1) as flow defined as:

$$\begin{aligned} x(t) &= x_0 e^{\lambda(\varepsilon)t} \\ y(t) &= y_0 e^{\mu(\varepsilon)t} \end{aligned} \tag{2}$$

And hence we can define the correspondence map as follows: $\Delta_\varepsilon^{\text{sing}} : \Gamma^+ \to T^-$

$$\Delta_\varepsilon^{\text{sing}}(y^{\alpha(\varepsilon)}, 1), \qquad \text{where } \alpha(\varepsilon) = -\frac{\lambda(\varepsilon)}{\mu(\varepsilon)} > 1$$

Poincare map and fixed point! *regular map:*

$$\Delta_\varepsilon^{\text{reg}} : T^- \to T^+$$
$$(x, 1) \mapsto (1, f_\varepsilon(x))\Delta$$

By the second genericity assumtion: $\partial f_\varepsilon(0)/\partial\varepsilon|_{\varepsilon=0} \neq 0$. WLOG assume (otherwise just change sign of parameter)

$$\frac{\partial f_\varepsilon(0)}{\partial \varepsilon}|_{\varepsilon=0} > 0$$

leading to following poincare map:

$$\Delta_\epsilon(y) = f_\varepsilon\left(y^{\alpha(\varepsilon)}\right)$$

2 HOMEWORK:

For the homework assignement of this week, I would like you to look at the final part of this proof: which means that you hae to show that the domain is an invariant set: The domain will be mapped to itself under the Poincaré map.
The we will have shown that the poincare map has a unique fixed point.
Hints:

- Use the second genericity assumttion
- Use that correspondence map has lipschitz constant
- define an arbitrary domain and show it will be mapped to itself under the poincare map

□

3 3D case

Now for the 3D case I will go through the most simple case: hyperbolic saddle with three real eigenvalues.

We may assume (WLOG) that the saddle has 2 negative eigenvalues 1 positive eigenvalue (otherwise reverse time) $\Rightarrow$

$\rightarrow$ has 2D stable manifold, and 1D unstable manifold

$\rightarrow$ In this case: Saddle value $=$ (maximum negativeλ) $+$ (positiveλ)

The Andronov-Leontovich theorem can be stated as follows:

Theorem 3.1. *Suppose that in a typical one-parameter family of smooth vector fields in $\mathbb{R}^3$, the zero value of the parameter corresponds to a 'critical' vector field with homoclinic orbit γ of a saddle having 2 negative eigenvalues and 1 positive eigenvalue. Then the vector fields corresponding to all sufficiently small values of the parameter on one side of zero having a hyperbolic periodic orbiot which tends to the homoclinic orbit γ of the critical vector field as the parameter tends to zero.*
stable periodic orbit if $\sigma < 0$
has 2D stable and unstable manifolds if $\sigma > 0$
Vector fields corresponding to all sufficiently small values of parameter on other side of zero have no periodic orbits in some neighborhood of γ

The genericity assumptions are now extended to 4 assumptions

1.2. Genericity assumptions. Before proving the theorem above, let us specify the genericity assumptions that the family of vector fields must satisfy.

1. *The eigenvalues of the hyperbolic saddle corresponding to the critical value of the parameter are nonresonant and pairwise disjoint.*

The above assumption implies that in some neighborhood of the saddle the vector field is smoothly equivalent to its linear part. Therefore, in the normalizing chart, its stable manifold is a plane and the unstable one is a line. On the stable plane, all phase curves except the singular point and two others tend to a singular point along the *leading stable direction* corresponding to the maximum negative eigenvalue as $t \to +\infty$.

2. *The homoclinic trajectory tends to the saddle along the leading stable direction as $t \to +\infty$.*

The next assumption makes use of the following remark. A saddle satisfying assumption 1 has an invariant plane W spanned by the eigenvectors associated to the two maximum eigenvalues. This invariant surface W may be prolonged along the homoclinic orbit. Now we can state the third assumption.

3. *The manifold W and the stable manifold of the saddle intersect transversally along the homoclinic orbit* (*see Figure* 7.2).

The next assumption is a property of the family itself.

4. *As the parameter passes through zero, the homoclinic orbit occurs and breaks in a transversal manner* (*a precise definition will be given below*).

In the proof of Theorem 1.1, only the assumptions 1–4 will be used.

Figure 1: Genericity assumptions 3D, 3 real eigenvalues

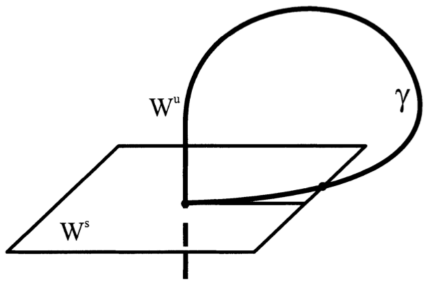


FIGURE 7.1. A hyperbolic saddle in the space, with real eigenvalues and a homoclinic loop.

4 Next lecture: Saddle-focus homoclinic bifurcation

Next lecture I will focus on **Saddle-focus homoclinic bifurcations**, which would mean that the leading stable eigenvalue is complex instead of real. The equilibrium in this case will be called a saddle-focus

a pair of complex eigenvalues in a planar system would correspond to either stable, unstable or non-hyperbolic equilibria, there would be no hyperbolic homoclinic bifurcations in that case. However in the 3-dimensional case, a pair of complex eigenvalues and a real eigenvalue give rise to a saddle-focus, to which homoclinic orbits may exist and undergo bifurcations.

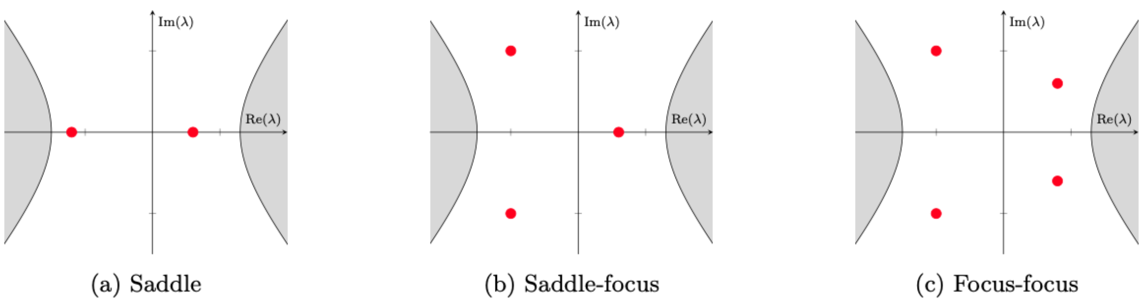


Figure 1.2: Configurations of leading eigenvalues λ (red). Gray area denotes non-leading eigenvalues.

As a final teaser, we can end up with something like: