# Lecture 2: Saddle-focus homoclinic bifurcation

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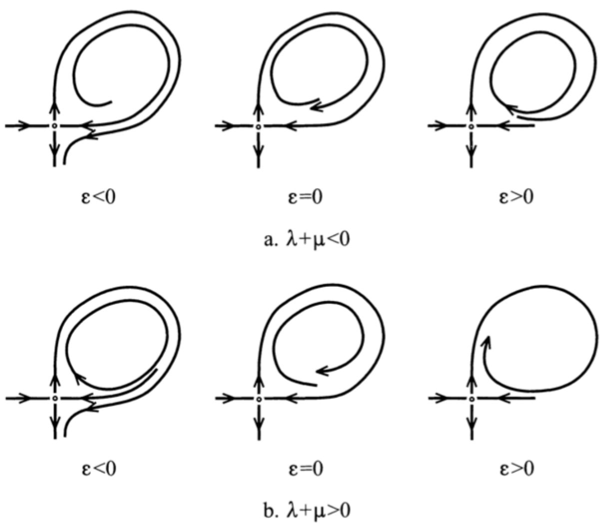
## 1 Overview of the lecture

1. Recap last lecture
2. 3D case saddle homoclinic case
3. saddle focus 3D case + connection to Horseshoe map
4. Introduce homework

## 2 Recap: last lecture

In the last lecture I looked at a saddle homoclinic bifurcation in the planar case (the 2D-case), and as part of homework you have finish the proof the Andronov-leontovich theorem for this planar case with the corresponding genericity assumptions. Here we used the poincare-map and that this can be decomposed in a regular and a singular (also correspondence map)
As a small recap:

- eigenvalues that have real parts of opposite signs (2 real eigenvalues)
- Saddle value: $\sigma = \lambda + \mu$
- one parameter family depending on $\varepsilon$: $X_\varepsilon$



These points were singular point of the first kind: homoclinicity to the singular points can be found on the boundary of morse-smale set.Unfoldings of these loops generate a unique cycle. Vector fields with homoclinic loops of points of the second kind never appear on the boundary of the Morse-Smale set. They have infinitely many periodic orbits, accumulating to the homoclinic loop.

**Definition 2.1.** *(Morse-Smale systems)*
A $C^1$ smooth vector field on a manifold (or diffeomorphism of this manifold) is called a Morse-Smale system is the following three conditions hold:

1. The set of nonwandering points consist of finite number of singular points and periodic orbits [for a diffeom.: a fixed point is periodic point of period 1]
   - Non-wandering: point in phase space is nonwandering for the flow if any small neighborhood of that point, when moved by the flow, intersects its original position in any distant future
2. All singular points and periodic points are hyperbolic
3. stable and unstable manifolds of hyperbolic singular points and periodic orbits intersect transversally (intersection either empty or at each intersection point the tangent spaces together span the tangent space to ambient manifold)

*Theorem:* Morse-Smale system on compact manifold are structurally stable

# 3 3D case: saddle homoclinic

Last time, I discussed the saddle homoclinic bifurcation in the planar case, including the difficult Andronov-leontovisch theorem. But we can also look at the 3D case. For the 3D case we have the following options for the eigenvalues:

- **Saddle case:** 3 real eigenvalues (2 neg, 1 pos)$\Rightarrow \sigma = \max \text{ neg}(\lambda) + \text{pos}(\lambda)$
- **Saddle-focus case:** 2 complex, 1 real (complex: neg real part, real pos) $\Rightarrow$ $\sigma = \text{Real}(\lambda_1) + \text{Real}(\lambda_{2,3})$ (The sum of the real part of the complex eigenvalues and the real one is the saddle value.)

For the most simple case we again assume that vector fields are homoclinic to fixed points that can be found on the boundary of the Morse-smale set

## 3.1 Saddle 3D

WLOG, we may assume that the saddle has 2 negative eigenvalues 1 positive eigenvalue (otherwise reverse time):
$\rightarrow$ has 2D stable manifold, and 1D unstable manifold
$\rightarrow$ In this case: Saddle value $=$ (maximum negative$\lambda$) + (positive$\lambda$)
The Andronov-Leontovich theorem can be stated as follows:

**Theorem 3.1.** *Suppose that in a typical one-parameter family of smooth vector fields in $\mathbb{R}^3$, the zero value of the parameter corresponds to a 'critical' vector field with homoclinic orbit $\gamma$ of a saddle having 2 negative eigenvalues and 1 positive eigenvalue. Then the vector fields corresponding to all sufficiently small values of the parameter on one side of zero having a hyperbolic periodic orbit which tends to the homoclinic orbit $\gamma$ of the critical vector field as the parameter tends to zero.*
*stable periodic orbit if $\sigma < 0$*
*has 2D stable and unstable manifolds if $\sigma > 0$*
*Vector fields corresponding to all sufficiently small values of parameter on other side of zero have no periodic orbits in some neighborhood of $\gamma$*

The genericity assumptions are now extended to 4 assumptions, that the family of vector fields must satisfy

1. the eigenvalues of the the hyperbolic saddle corresponding to the critical value of the parameter are nonresonant and pairwise disjoint

   implication: in some neighborhood of the saddle the vector field is smoothly equivalent to its linear part. Therefore, in the normalizing chart, its stable manifold is a plane and the unstable one is a line.

   On stable plane: all phase curves, except singular point and two others, tend to singular point along the leading stable direction corresponding to maximum negative eigenvalue as $t \to +\infty$

2. the homoclinic trajectory tends to the saddle along the leading stable direction as $t \to +\infty$

3. the manifold $W$ and stable manifold of the saddle intersect transversally along homoclinic orbit as in Figure 2

4. (FOR FAMILY ITSELF) As the parameter passes through zero, the homoclinic orit occurs and breaks in transversal manner

   *meaning:* the image $O(\varepsilon) = \Delta_\varepsilon^{\text{reg}}(O)$ transversallt intersects the line $z = 0$ as $\varepsilon$ passes through zero.

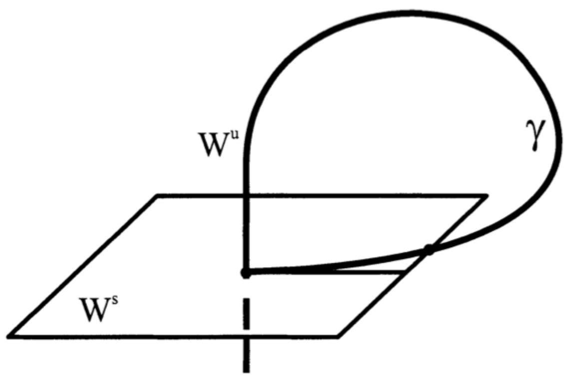



FIGURE 7.1. A hyperbolic saddle in the space, with real eigenvalues and a homoclinic loop.

Figure 1: Such a saddle has a 2D stable ($W^s$) and a 1D unstable manifold ($W^u$)

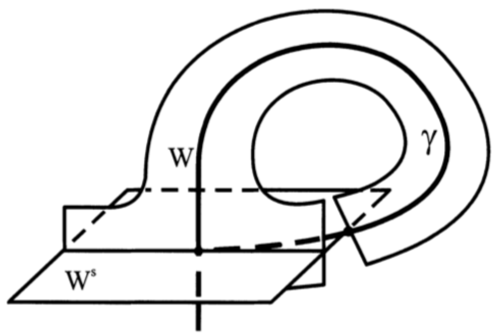



FIGURE 7.2. Transversality assumption for the homoclinic orbit of the saddle.

Figure 2: Intersect 3D

### 3.2 Outline proof:

The whole proof in the book takes about 7 pages, using again vertically and horizontally cylindrical rectangles, the hyperbolic fixed point theorem and the cone condition for the poincare map. So I will not go over the details of this proof; you can find in in paragraphs 7.1 of the book on nonlocal bifurcation if you are interested: I will mention some properties used in this proof
We again have poincare map split into two parts: $\Delta_\varepsilon = \Delta_\varepsilon^{\text{reg}} \circ \Delta_\varepsilon^{\text{sing}}$ with cross-sections $\Gamma^+$ and $\Gamma^-$ close to the singular point and transversal to the homoclinic orbit
*Singular map:* can be made explicit using theory of normal forms: $\Delta_\varepsilon^{\text{sing}} : \Gamma^+ \to \Gamma^-$
genericity assumption 1 + theory of finitely smooth normal forms for local families $\Rightarrow$ with

$$\Gamma^+ = \{x = 1, |y| \leq 1, |z| \leq 1\}$$
$$\Gamma^- = \{z = 1, |x| \leq 1, |y| \leq 1\}$$

we can define the projection of any orbit along z-axis to the $(x,y)$-plane as the orbit of the following system

$$\begin{cases} \dot{x} = \lambda_1(\varepsilon)x, & \lambda_1(0) < 0 \\ \dot{y} = \lambda_2(\varepsilon)y, & \lambda_2(0) < \lambda_1(0) \end{cases}$$

The singular map is a strong contraction in the y-direction. with the definition of the saddle value $\sigma$, we can have two cases:

- $\sigma > 0 \Rightarrow$ instability wins $\Rightarrow$ hyperbolic, strong contraction (expansion) in '$y$'-direction (orthogonal direction), with contraction/expansion coefficient converging towards $\infty$
- $\sigma < 0 \Rightarrow$ stability wins $\Rightarrow$ strong contraction (for small $z$), contraction coeff converging towards 0

For the singular correspondence map, we have the following equation:

$$\Delta_\varepsilon^{\text{sing}}(y, z) = (yz^\alpha, z^\beta), \qquad \text{with } \alpha = \alpha(\varepsilon) = \frac{\lambda_2(\varepsilon)}{\mu(\varepsilon} > \beta(\varepsilon), \qquad \text{with } 0 < \mu(0)$$

*Regular map:* smooth with respect to the phase variables, which are coordinates on $\Gamma^-$, and the parameters. This map needs to be as simple as possible to describe the geometric effects. The following form of $\Delta_\varepsilon^{\text{reg}}$ is assumed:

$$\Delta_\varepsilon^{\text{reg}} = g_\varepsilon^\pm \quad \text{with}$$
$$g_\varepsilon^+ : (x, y, 1) \to (1, y, x + \varepsilon) \qquad \text{orientable}$$
$$g_\varepsilon^- : (x, y, 1) \to (1, -y, -x + \varepsilon) \qquad \text{NON-orientable}$$

*Poincare map:*

$$\Delta_\varepsilon = \Delta_\varepsilon^{\text{reg}} \circ \Delta_\varepsilon^{\text{sing}}$$
$$\Delta_\varepsilon^{\text{reg}} = g_\varepsilon^{\pm}$$
$$\Delta_\varepsilon^{\text{sing}}(y,z) = (yz^\alpha, z^\beta)$$

# 4 Saddle-focus 3D

Now that we have discussed the most simple version of the 3D-case, we can also discuss a more interesting case, where we assume that we have a pair of complex eigenvalues and a real eigenvalue, giving rise to the SADDLE-FOCUS.
This part was quite complicated in the nonlinear bifurcations book, so for this part I have used a different source. But if you would like you can find a proof in the same manner as the previous one in paragraph 7.2 in the book: Nonlinear bifurcations by Ilyashekno. In this book the link to the smale horseshoe is explained, with the cone condition, a whole lot of in between defined maps and horizontal/vertically cylindrical rectangles.
To these saddle-foci, homoclinic orbits way exist and can undergo bifurcations. This can be captured in the following theorem:

**Theorem 4.1.** *(shilikov's thm)*
*Consider a smooth function $f$, satisfying:*

$$\dot{x} = f(x,\alpha), \quad s \in \mathbb{R}^3, \alpha \in \mathbb{R},$$

*Now, let us assume that this system has a saddle-focus equilibrium at $O = 0$ with a pair of complex eigenvalues and 1 real eigenvalue satisfying:*

$$\lambda_1(x_0) > 0,$$
$$\mathbb{R}(\lambda_{2,3}) < 0,$$

*from which we can define the saddle value as follows:*

$$\sigma = \lambda_1(O) + Re(\lambda_{2,3}(O))$$

*Furthermore we have a homoclinic orbit $\gamma$.*
*We can look at the following two cases:*

- $\sigma < 0 \Rightarrow$ *homoclinic loop of the cycle generates a stable limit cycle (same as in case with three real eigenvalues) (first kind)*
- $\sigma > 0 \Rightarrow$ *unperturbed vector field has complicated invariant set, system had $\infty$ number of saddle limit cycles in a neighborhood of $\gamma \cup O$, implying the existence of an infinite set of horseshoes (these cannot occur on the boundary of Morse-Smale set, hence we have of the second kind)*

The first case is similar to the previous discusses 3D saddle case (with 3 real eigenvalues), but the more wild case, giving a link to smale-horseshoe maps is a lot more interesting.
The geometric construction looks as follows, and I will fill in the blanks while I am going through the mathematical details of the proof:

*Proof.* We consider the following 3D system:

$$\begin{cases} \dot{x} = \text{Re}(\lambda_{2,3})x + \text{Im}(\lambda_{2,3})y + f_1(x,y,z), \\ \dot{y} = -\text{Im}(\lambda_{2,3})x + \text{Re}(\lambda_{2,3})y + f_2(x,y,z), \\ \dot{z} = \lambda_1 z + f_3(x,y,z) \end{cases} \tag{1}$$

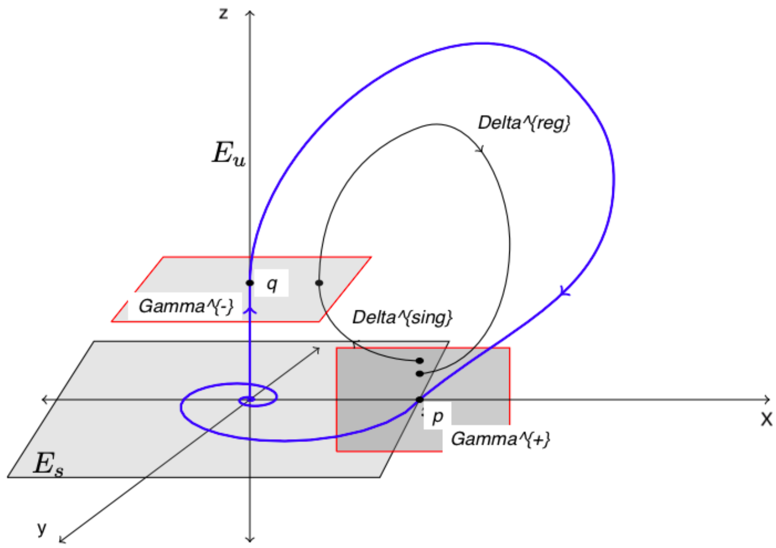



Figure 3: Geometric construction of proof for saddle-focus

The Taylor expansions of $f_i$ have a zero linear part. We furthermore assume that the homoclinic orbit passes through (0,0,1) and (1,0,0). Near singular point $O$, we consider the following linear system:

$$\begin{cases} \dot{x} = \mathrm{Re}(\lambda_{2,3})x + \mathrm{Im}(\lambda_{2,3})y \\ \dot{y} = -\mathrm{Im}(\lambda_{2,3})x + \mathrm{Re}(\lambda_{2,3})y, \\ \dot{z} = \lambda_1 z. \end{cases}$$

We can do this since the flow of the system in Equation (1) is $C^1$ equivalent near the saddle-focus to the flow of the linearization.
Now we construct to cross-sections; $\Gamma^+$, $\Gamma^-$, close to the saddle focus and such that $\Gamma^+$ is transversal to the stable manifold, while $\Gamma^-$ is transversal to the unstable manifold. We define the cross-sections as:

$$\Gamma^+ = \{(x, y, z)|y = 0\}$$
$$\Gamma^- = \{(x, y, z)|z = 1\}$$

We define the poincare map: $\Delta : \Gamma^+ \to \Gamma^+$ as a composition of two maps: $\Delta = \Delta^{\mathrm{reg}} \circ \Delta^{\mathrm{sing}}$, with:

$$\Delta^{\mathrm{sing}} : \Gamma^+ \to \Gamma^-$$
$$\Delta^{\mathrm{reg}} : \Gamma^- \to \Gamma^+$$

Now we consider the following two points as in the figure:

$$p = (x_p, 0, z_p),$$
$$q = (x_q, y_q, 1),$$

which we can use to define the singular map:

$$\Delta^{\mathrm{sing}} : \begin{bmatrix} x_p \\ z_p \end{bmatrix} \mapsto \begin{bmatrix} x_p(z_p)^\nu \cos\left(-\frac{\mathrm{Im}(\lambda_{2,3})}{\lambda_1}\ln(z_p)\right) \\ x_p(z_p)^\nu \cos\left(-\frac{\mathrm{Im}(\lambda_{2,3})}{\lambda_1}\ln(z_p)\right) \end{bmatrix},$$

where $\nu = -\frac{\mathrm{Re}(\lambda_{2,3})}{\lambda_1}$, which is called the saddle index.

Now the regular map: $\Delta^{\text{reg}} : \Gamma^- \to \Gamma^+$ is in general taken as a $C^1$ map. and is formally mapping $q$ as follows:

$$\Delta^{\text{reg}} : \begin{bmatrix} x_q \\ y_q \end{bmatrix} \mapsto \begin{bmatrix} 1 + ax_q + by_q \\ \mu + cx_q + dy_q \end{bmatrix} + \mathcal{O}(\|q\|^2). \tag{2}$$

In order to guarantee local invertibility we have a restriction on $(a, b, c, d)$: $ad - bc \neq 0$.
Finally, we can explicitly define the Poincare map:

$$\Delta : \begin{bmatrix} x_p \\ z_p \end{bmatrix} \mapsto \begin{bmatrix} 1 + Ax_p(z_p)^\nu \sin\left(-\frac{\text{Im}(\lambda_{2,3})}{\lambda_1}\ln(z_p)\right) \\ \mu + B_p(z_p)^\nu \sin\left(-\frac{\text{Im}(\lambda_{2,3})}{\lambda_1}\ln(z_p)\right) \end{bmatrix} + \mathcal{O}(\|p\|^2),$$

mapping $\Gamma^+$ to itself. The fixed point of the Poincare map reveal that the bifurcations occurs in small neighborhood of $\gamma \cup O$, hence the fixed point has to satisfy that the input $=$ output.
, by some rewrite we can find:

$$x = \mu + x^\nu \sin\left(-\frac{\text{Im}(\lambda_{2,3})}{\lambda_1}\ln(x)\right)$$

where at the same time the higher order terms are dropped since the goal is to observe small $\|x\|$ effects. Here the value $\mu$ can be used to shift the function $F$, which will be defined later. This will influence whether or not there will be more than one fixed point for the case $\nu > 1$. This latter equation is the scalar fixed point condition for the saddle-focus case.
this condition can be written by defining a map: $F(x, \mu)$

$$F : x \mapsto \mu + x^{nu} \sin\left(-\frac{\text{Im}(\lambda_{2,3})}{\lambda_1}\ln(x)\right)$$

The following can now be observed

- $\nu < 1$ $(\sigma > 0)$: infinitely many fixed points exist
- $\nu > 1$ $(\sigma < 0)$: finitely many (at least 1) fixed points exist for all sufficiently small values of $\mu$

These can be visualized as follows:

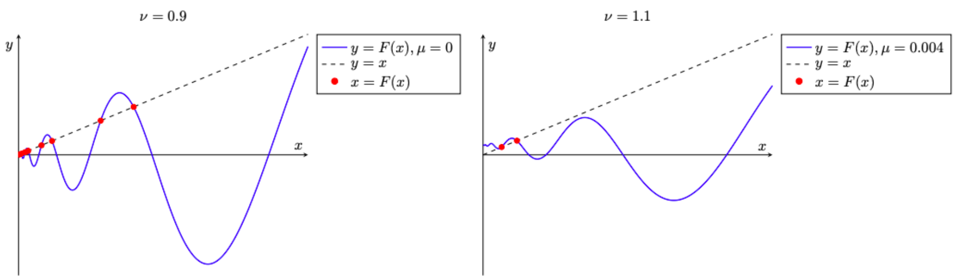



Figure 2.11: Plots of the function (2.21) for $\nu < 1$ and $\nu > 1$. The parameter $\mu$ shifts the curve up (down) for positive (negative) values. In the case $\nu < 1$, we see that there exist infinitely many fixed points (and thus periodic orbits) at $\mu = 0$ (the homoclinic orbit) for small values of $x$. For $|\mu|$ sufficiently small, the infinitely many fixed points persist. In the case $\nu > 1$, for $\mu = 0$ the only fixed point is $x = 0$. For $|\mu|$ and $x > 0$ sufficiently small,it is possible to see finitely many more fixed points, or none at all.

Figure 4: visualization of number fixed points

□

## 4.1 Link to horseshoe

Now the smale horseshoe existence thm states has the assumption that a map, which is a diffeormorphism, mapping a union of horizontally cylincdric $(\mu_h, \mu_v)$-rectangles to a subset of a standard rectangle (for which I discussed the definition last time) to satisfy the $(\mu_h, \mu_v)$ cone condition, which can be proven.

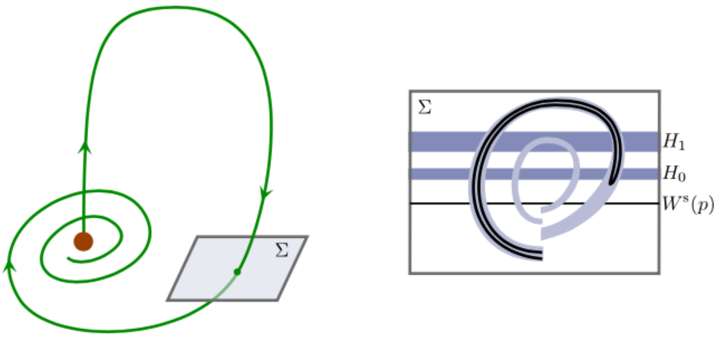



Figure 5.2: *A Shil'nikov saddle-focus homoclinic orbit is shown in the left panel. If the complex eigenvalues are closest to the imaginary axis, then the return map* $\Pi : \Sigma \to \Sigma$ *contains infinitely many horseshoe maps obtained by restriction to strips* $H_i$. *The second iterate* $\Pi^2$ *restricted to a union* $H_i \cup H_j$ *may contain nonhyperbolic dynamics: this is illustrated in the right panel where the horseshoe-shaped images of two strips under* $\Pi$ *and, with a darker shade, part of the image under* $\Pi^2$ *of the union of the strips are shown.*

Figure 5: horseshoe

# 5 Homework

For the homework assignment I searched for an article that is about saddle-foci in a more apllicational form and the arising chaos in a specific system. In the last thirty years, many authors have tried to find the mathematically simplest systems of various species that can exhibit chaos. For a long period of time the Lorenz and Rössler systems were regarded as the simplest chaotic autonomous dissipative systems of ODEs. And in the paper, that I have found,the so-called Rössler prototype-4 system is considered.
*ASSIGNMENT:*
Try to summarize the article (introduction, part on global analytical analysis and numerical investga-tion) and try to keep the focus on the saddle-focus fixed points.
I will update these notes and the pdf file of the article on the website. In this pdf I will highlight the paragraphs to focus on.

# 6 References


- Chapter 7.1 and 7.2 of Nonlinear Bifurcations, written by Yu. Ilyashenko and Weigu Li
- Master thesis titled: 'Homoclinic saddle to saddle-focus transitions in 4D systems' written by Manu Kalia in 2017