The Hamiltonian period-doubling bifurcation

the plan of the presentation
- Flouquet exponents / multipliers
- pitchfork bifurcation
- flip bifurcation
- period-doubling bifurcation

Consider the Hamiltonian $H(x,y,q,p) = \omega y + (\alpha + 2\pi\omega)\frac{p^2+q^2}{2}$, $\quad x \in T = \mathbb{R}/2\pi\mathbb{Z}$

with the symplectic structure: $dx \wedge dy + dq \wedge dp$.

choosing the co-ordinates x and y wisely makes ω independent of x and passing to Floquet co-ordinates q and p makes α independent of x (the former always works and the latter works unless a Floquet multiplier is equal to -1)

the corresponding equations of motion:

$$\dot{x} = \frac{\partial H}{\partial y} = \omega$$

$$\dot{y} = -\frac{\partial H}{\partial x} = 0$$

$$\dot{q} = \frac{\partial H}{\partial p} = (\alpha + 2\pi\omega)p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -(\alpha + 2\pi\omega)q$$

$$\begin{pmatrix} 0 & \alpha+2\pi\omega \\ -(\alpha+2\pi\omega) & 0 \end{pmatrix}$$

eigenvalues: $\pm(\alpha i + 2\pi i\omega)$ → Floquet exponents

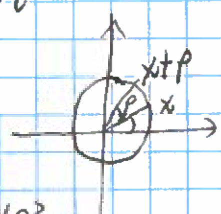
$e^{\pm(\alpha i + 2\pi i \omega)}$ → Floquet multipliers

pitchfork bifurcation:

phase space $T^1 \times \mathbb{R} \times \mathbb{R}^2$

$$H(x,y,q,p) = \omega y + \frac{1}{2}p^2 + \frac{1}{24}q^4 + \frac{1}{2}yq^2$$

S^1-symmetry: $x \mapsto x + \varphi$



S^1-symmetry ↓ $\{y\} \times \mathbb{R}^2$

$$H(x+\varphi, y, q, p) = \omega y + \frac{1}{2}p^2 + \frac{1}{24}q^4 + \frac{1}{2}yq^2$$

↓ a fixed value of parameter

x: cyclic angle $\Rightarrow$ y: is an integral of motion

($\dot{y} = 0 \Rightarrow y = \text{constant}$)

y is conserved.

we can see y as a parameter

$\mathcal{H}_y(q,p) = \frac{1}{2}p^2 + \frac{1}{24}q^4 + \frac{1}{2}yq^2$

reduce form (becomes simpler)

2 degree-of freedom $\longrightarrow$ 1 d.o.f.

4-dimsional $\longrightarrow$ 2-dimsional

the corresponding equations of motion:

I: equilibria

$\dot{q} = \frac{\partial \mathcal{H}_y}{\partial p} = p \qquad = 0 \qquad p = 0$

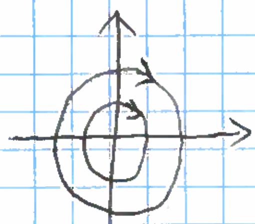
$\dot{p} = -\frac{\partial \mathcal{H}_y}{\partial q} = -\frac{q^3}{6} - yq \qquad = 0 \qquad q = 0, \pm\sqrt{-6y}$

$\begin{pmatrix} 0 & 1 \\ -\frac{q^2}{2}-y & 0 \end{pmatrix} \Bigg|_{(q,p)=(0,0),\,(\pm\sqrt{-6y},\,0)}.$

$y \in \mathbb{R} \quad \begin{cases} y<0 \\ y=0 \\ y>0 \end{cases}$

$y>0$: at $(q,p)=(0,0)$,

$\begin{pmatrix} 0 & 1 \\ -y & 0 \end{pmatrix}$ eigenvalues: imaginary



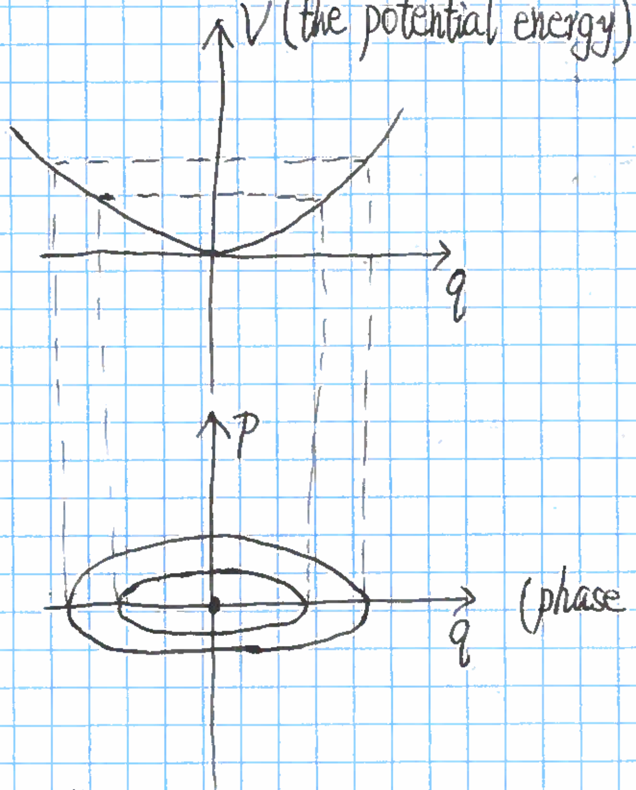
centre

at $(q,p) = (\pm\sqrt{-6y}, 0)$,

because $y>0$, $\pm\sqrt{-6y}$ is not real value $\Rightarrow$ there is no equilibrium.

$y=0$: at $(q,p) = (0,0)$,

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ parabolic



(phase potrait)

$\mathcal{H}_y(q,p) = \underbrace{\frac{1}{2}p^2}_{\text{kinetic}} + \underbrace{\frac{1}{24}q^4 + \frac{1}{2}yq^2}_{\text{the potential energy}}$

because $y=0$,

here $V = \frac{1}{24}q^4$

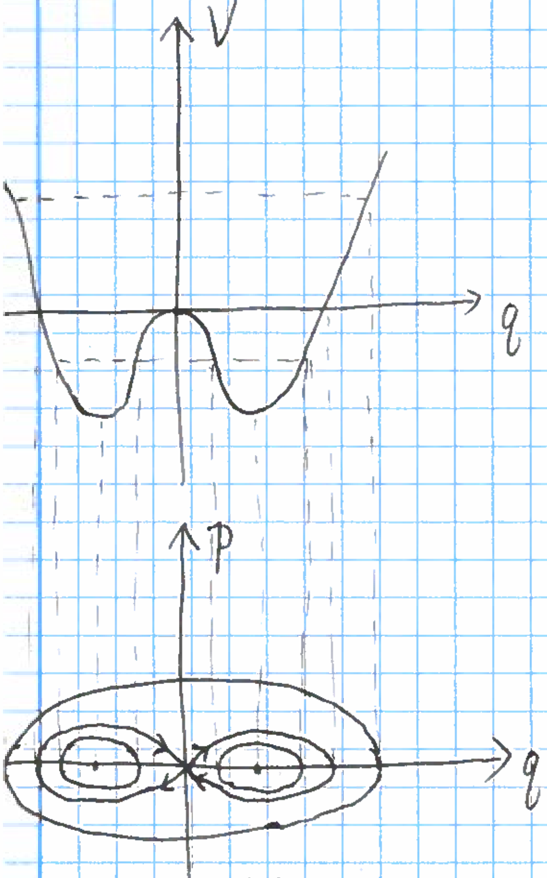
at $(q,p) = (\pm\sqrt{-6y}, 0)\big|_{y=0} = (0,0)$,

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ parabolic the same as above.

0: at $(q,p)=(0,0)$,

$\begin{pmatrix} 0 & 1 \\ |y| & 0 \end{pmatrix}$ eigenvalues: $\pm\sqrt{|y|}$ saddle

$$V=\underbrace{\frac{1}{24}q^4}_{V_1}+\underbrace{\frac{1}{2}yq^2}_{V_2}$$



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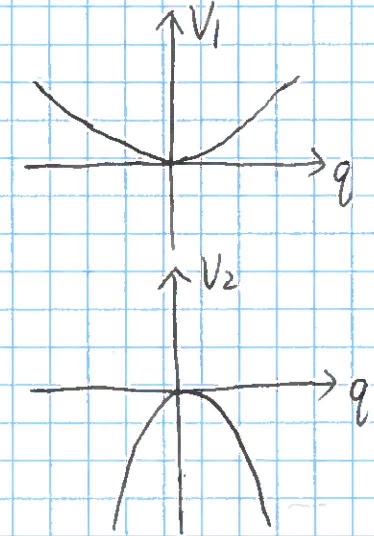
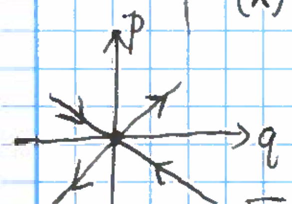


$q\to 0$: $|V_1|<|V_2|$ $V<0$

q large: $|V_1|>|V_2|$ $V>0$



$e^{\sqrt{|y|}t}\to\infty$ $t\to\infty$ $e^{-\sqrt{|y|}t}\to 0$ $t\to\infty$

t $(q,p)=(\pm\sqrt{-6y},0)$,

$\begin{pmatrix} 0 & 1 \\ 2y & 0 \end{pmatrix}$ eigenvalues: imaginary centres

he phase potrait is the same as (*).

$\mathbb{Z}_2$- symmetry: $(q,p)\longmapsto(-q,-p)$

,p)-plane π-rotation

the symplectic structure is invariant: $dq\wedge dp\longmapsto dq\wedge dp$

$\mathcal{H}_y(-q,-p)=\frac{1}{2}p^2+\frac{1}{24}q^4+\frac{1}{2}yq^2$

this means the Hamiltonian $\mathcal{H}_y(q,p)$ doesn't change. $\dim(\mathbb{Z}_2)=0$.

Why do we do $\mathbb{Z}_2$-symmetry?

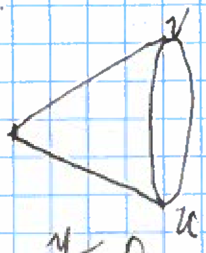
choose a basis of invariants: $u=\frac{q^2}{2}$, $v=\frac{p^2}{2}$, $w=pq$.

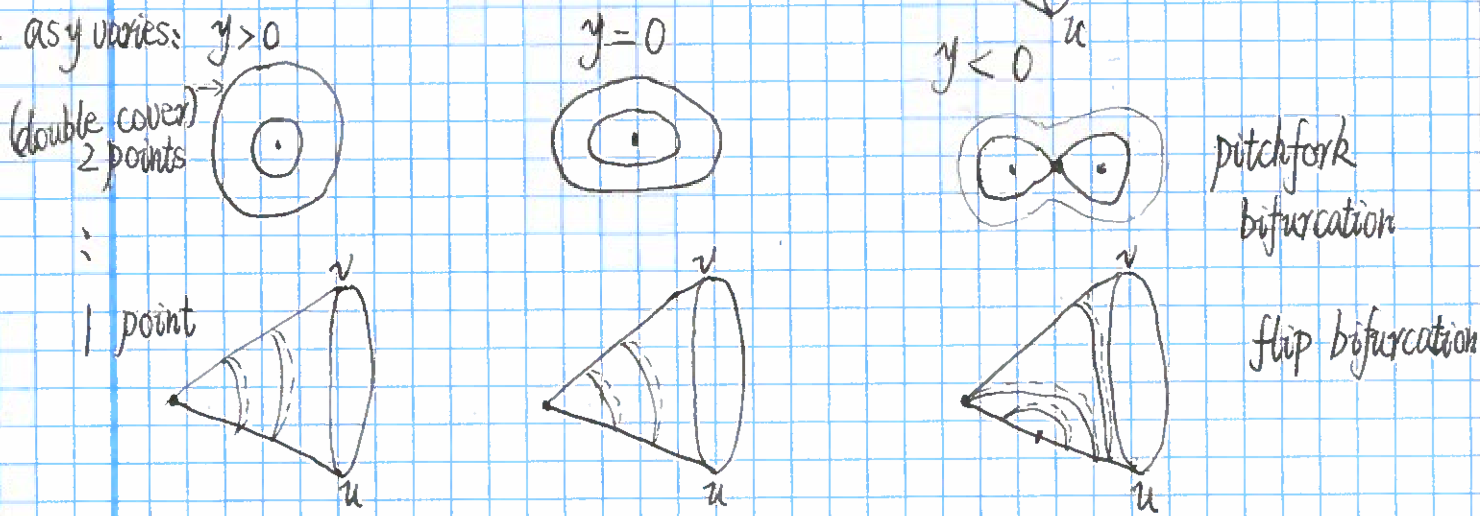
$$\mathcal{H}_y(u,v,w)=v+\frac{1}{6}u^2+yu.$$

the order of $\mathcal{H}_y(q,p)$ is 4, the order of $\mathcal{H}_y(u,v,w)$ is 2.

~~bec~~ to some extent: becomes simpler.

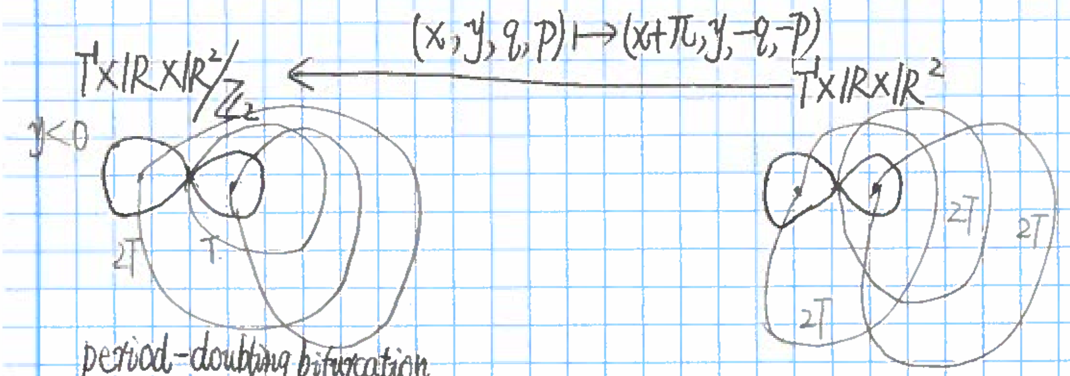
cone: $2uv=\frac{w^2}{2}$, $u\geq 0$, $v\geq 0$

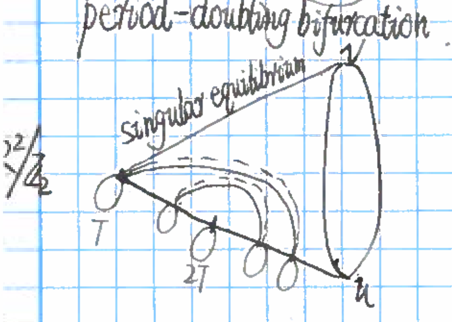




After the reduction of the $\mathbb{Z}_2$-symmetry, the pitchfork bifurcation becomes flip bifurcation.

$$\begin{CD}
T^1\times\mathbb{R}\times\mathbb{R}^2/\mathbb{Z}_2 @<{(x,y,q,p)\mapsto(x+\pi,y,-q,-p)}<< T^1\times\mathbb{R}\times\mathbb{R}^2\\
@V{S^1\text{-symmetry}}VV @VV{S^1\text{-symmetry}}V\\
\{y\}\times\mathbb{R}^2/\mathbb{Z}_2 @<{(q,p)\mapsto(-q,-p)}<< \{y\}\times\mathbb{R}^2
\end{CD}$$





the others: $T^2=T^1\times T^1$

Up to above, all are examples.

The book [Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems] on page 98:

Theorem (general statement):

Let H define a Hamiltonian system in two degrees of freedom and let γ be a period orbit of X_H with Floquet multiplier -1. Under generic conditions on H a period-doubling bifurcation takes place as the value of H passes through $H(\gamma)$.

Explanation: Here, $H: T^1\times\mathbb{R}\times\mathbb{R}^2/\mathbb{Z}_2\to\mathbb{R}$ has a period orbit with Floquet multiplier -1. So for Floquet co-ordinates, they have to pass to a double cover space. $T^1\times\mathbb{R}\times\mathbb{R}^2$ is a double cover of $T^1\times\mathbb{R}\times\mathbb{R}^2/\mathbb{Z}_2$, on the phase space $T^1\times\mathbb{R}\times\mathbb{R}^2$, around the period orbit $T^1\times\{0\}\times\{0,0\}$, we can choose Floquet co-ordinates (x,y,q,p), for example, there have $\hat{H}(x,y,q,p)=\omega y+\frac{p^2}{2}+\frac{q^4}{24}+\frac{yq^2}{2}$ defined on $T^1\times\mathbb{R}\times\mathbb{R}^2$, then by reducing the S^1-symmetry, we get $\hat{H}_y(q,p)=\frac{p^2}{2}+\frac{q^4}{24}+\frac{yq^2}{2}$. After reducing the $\mathbb{Z}_2$-symmetry $\mathcal{H}_y(u,v,w)=v+\frac{1}{6}u^2+yu$, then the dynamics on $\{y\}\times\mathbb{R}^3/\mathbb{Z}_2$ are clear, so the dynamics on $T^1\times\mathbb{R}\times\mathbb{R}^2/\mathbb{Z}_2$ are clear.

Exercise:

Consider the Hamiltonian $H(x,y,q,p) = \omega y + \frac{1}{2}p^2 - \frac{1}{24}q^4 + \frac{1}{2}yq^2$ on $T^1 \times \mathbb{R} \times \mathbb{R}^2$. Follow this programme to explain the dual case of the period-doubling bifurcation.

— reduce the S^1-symmetry

— reduce the $\mathbb{Z}_2$-symmetry

— explain how it is related to a period-doubling bifurcation on $T^1 \times \mathbb{R} \times \mathbb{R}^2 / \mathbb{Z}_2$.

— concentrate on the differences to the previous ~~case~~ case.

(students can be short on what works in exactly the same way)