Umbilical bifurcations of equilibria with zero linear part

Topics:

(1) Introduction/Recap

(2) Linear classification

(3) Description of the elliptic umbilic catastrophe

(4) —— " —— hyperbolic —— " ——

(1) Introduction

Aim

Describe generic unfoldings of this kind of equilibrium

Its codimension depends on whether the linear part is semisimple or not

every T-invariant subspace has a complementary —"—
$\Leftrightarrow$ minimal polyn. being square free

semisimple $\rightarrow$ codimension 3

non-semisimple $\rightarrow$ codimension 1

Remarks

(1) Although in the original family only generic restrictions are imposed on the lower order terms, (by normal form theory and singularity theory), the corresponding unfolding is reduced to an

arbitrarily flat perturbation of a normal form, completely determined by 1 d.o.f. system (polynomial of degree 3)

Recap

$H: \mathbb{R}^2 \to \mathbb{R}$ C^∞ functions with the origin as a critical point

$\overset{\text{induce}}{\rightsquigarrow}$ X_H: corresponding Hamiltonian vector field

$dH = \omega(X_H, \cdot)$, where $\omega = dx \wedge dy$: the standard area 2-form on $\mathbb{R}^2$

In coordinates: $\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases}$

Remark

For a vector field X on $\mathbb{R}^2$ to be hamiltonian w.r.t. H, X must have $\operatorname{div} X = 0$

(2) Linear classification

Consider the linear part

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix}$, where L is a 2×2 matrix

Since $\operatorname{div} X_H = 0$, we have that $\operatorname{Tr}(L) = 0 \Rightarrow L \in \mathfrak{sl}(2, \mathbb{R})$

Remark

If H has at the origin Hessian matrix $D_0^2 H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

then we have $L = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$

Escaneado con CamScanner

Now, consider the 3-dimensional space $sl(2,\mathbb{R})$ and define the following relation:

$L, H \in sl(2,\mathbb{R})$, $L \sim H \iff \exists S \in Gl(2,\mathbb{R})$, $\exists k \in \mathbb{R} \setminus \{0\} : S \circ L \circ S^{-1} = kH$

Remark

It's an equivalence relation.

Now if $L \in sl(2,\mathbb{R})$, $\chi_L(\lambda) = \lambda^2 + \det(L) \Rightarrow$

leads to the following partition in equivalence classes:

(i) hyperbolic case $\det(L) < 0$

$\rightsquigarrow$ corresponding normal form : $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $H(x,y) = xy$

(ii) elliptic case , $\det(L) > 0$

$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H(x,y) = \frac{1}{2}(x^2 + y^2)$

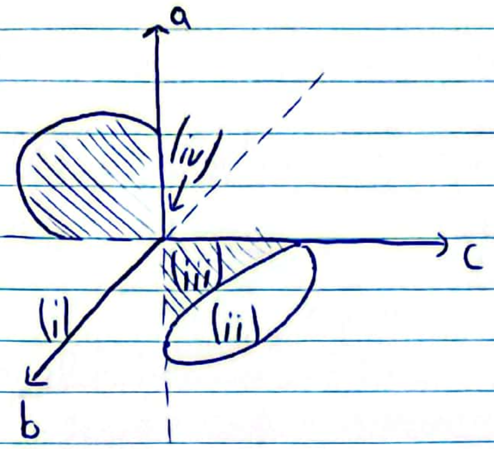
(iii) parabolic case , $\det(L) = 0$ and $L \neq 0$

$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H(x,y) = \frac{1}{2} y^2$

(iv) zero case , $L = 0$

$L = 0$, $H(x,y) \equiv 0$

Stratification of $sl(2,\mathbb{R})$



What happens to the above (linear) picture, if higher order terms are added?

(a) cases (i) and (ii) (codim = 0) can be handled by Morse lemma:

→ Let u be a nondegenerate critical point of the smooth function $f: \mathbb{R}^n \to \mathbb{R}$.
⟹ $\exists$ local coordinate system $(y_1, \dots, y_n)$ in a neighborhood U of u with $y_i(u) = 0$ $\forall i$, such that:
$f = f(u) - y_1^2 - \dots - y_i^2 + y_{i+1}^2 + \dots + y_n^2$, $\forall y \in U$

This implies that the normal forms given before also hold with the higher order terms added.

(* oral addition)

(b) In case (iii) the answer is given by the Splitting lemma:

→ Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $Df(0) = 0$, whose Hessian at 0 has rank r (and corank $n-r$). Then, around 0, f is equivalent to a function of the form:
$\pm x_1^2 \pm \dots \pm x_r^2 + \hat{f}(x_{r+1}, \dots, x_n)$, where $\hat{f}: \mathbb{R}^{n-r} \to \mathbb{R}$ is smooth

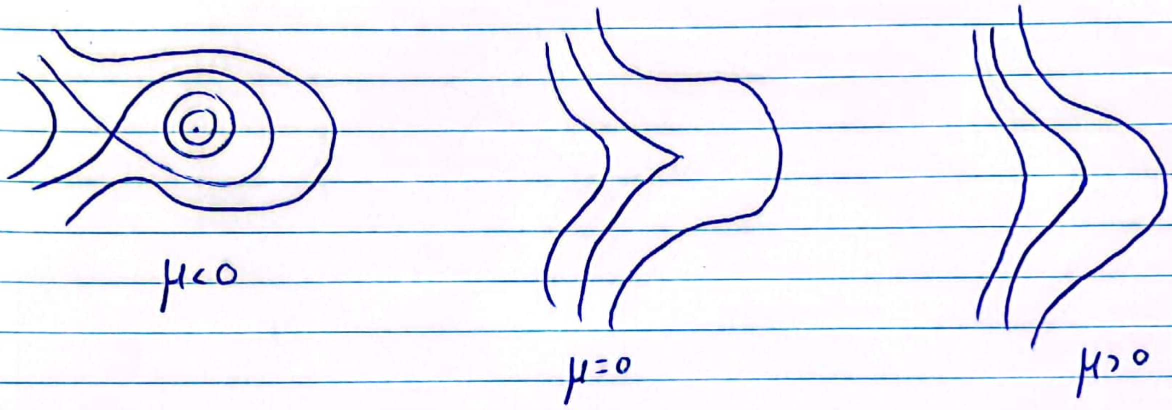
This tells us that we only need to consider 1-parameter families of the form:

$$H^{\mu}(x,y) = \frac{1}{2}y^2 + V^{\mu}(x) \quad , \text{ where } V^{0}(x) = O(x^3) \quad \text{as } x \to 0$$

Universal unfolding : $H^{\mu}(x,y) = \frac{1}{2}y^3 + \underbrace{\frac{1}{3}x^3 + \mu x}_{\text{Fold Catastrophe}}$

$\mu > 0$: no singularities exist
$\mu < 0$: saddle-point and center are present
$\mu = 0$: origin occurs as the new parabolic singularity

This is the Saddle-Center bifurcation.



(c) In case (iv) we don't get a convenient splitting and we have two universal unfoldings

$$H^{\mu,\nu,k}(x,y) = x^2y \pm \frac{1}{3}y^2 + \mu(x^2 \mp y^2) + \nu x + ky$$

upper sign $\longrightarrow$ hyperbolic umbilic catastrophe
lower sign $\longrightarrow$ elliptic — " —

(3)

Description of the elliptic umbilic catastrophe

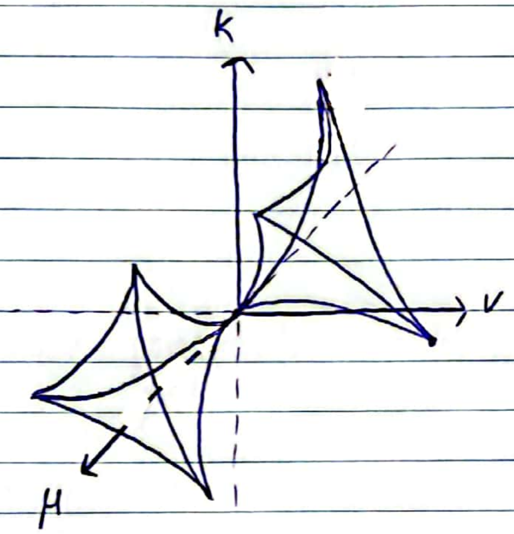
Catastrophe set $\mathcal{C} = \{(\mu, \nu, \kappa) \mid H^{\mu,\nu,\kappa}$ has a $\underbrace{\text{degenerate}}_{}$ critical point$\}$

Jacobian evaluated at this point is zero

It is given parametrically by:

$$\begin{cases} \mu^2 = x^2 + y^2 \\ \nu = -2\mu x - 2xy \\ \kappa = -2\mu y + y^2 - x^2 \end{cases}$$

, where x, y : viewed as parameters



Remarks

(i) Cone-like structure with a curvilinear triangle as above, the edges of which meet in cusps.

(ii) The axis of the "cone" is the μ-axis and the triangle shrinks quadratically in $|\mu|$ as $\mu \to 0$

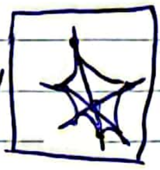
(iii) For $\mu \neq 0$ the parameter μ only effects the characteristic size of the phase-portraits, while its sign governs the orientation of the flow.

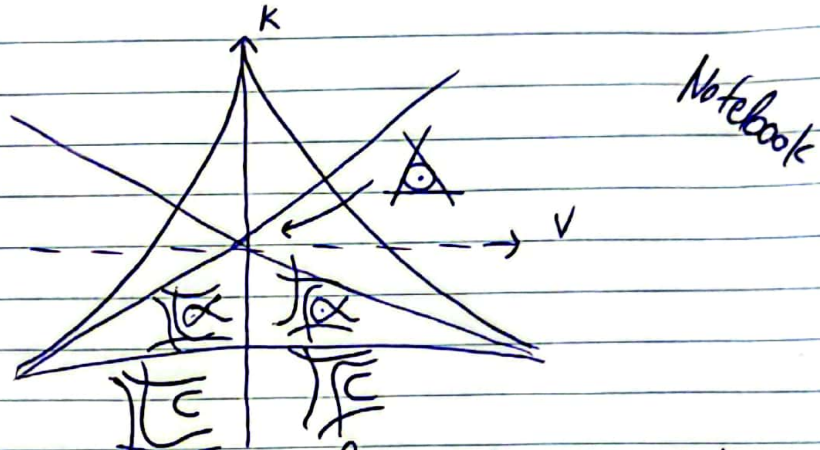
(iv) $\mu = 0$: central, umbilical singularity occurs at the origin

(v) 'Around the origin' the potential $V^{\mu,\nu}$ undergoes a <u>dual cusp catastrophe</u> and the corresponding Hamiltonian family has normal form: $H^{\mu,\nu}(x,y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}\mu x^2 + \nu x$

Bifurcation diagram of the elliptic umbilic for $\mu \neq 0$, fixed

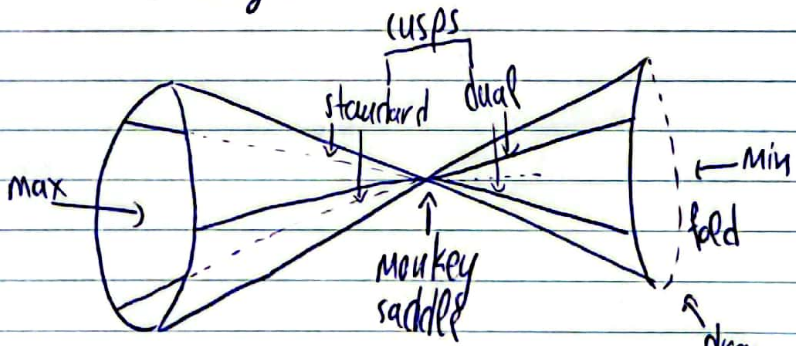
central singularity





Comments

- The phase-portraits are given relative to the 2-dimensional section $\mu = 1$
- We restricted to one sector of the diagram. The other phase portraits follow by symmetry
- The central umbilical singularity:
  - occurs at $(\mu, \nu, k) = 0$
  - corresponding critical point for $H^{0,0,0}(x,y) = x^2 y - \frac{1}{3}y^2$ is called <u>Monkey saddle</u>

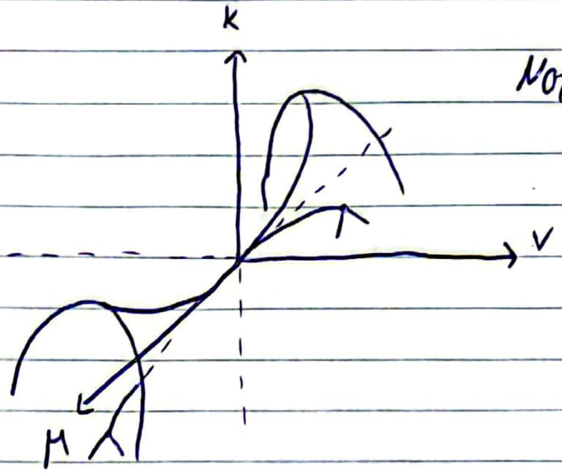


$H^{\mu,\nu,\kappa}(x,y) = \frac{1}{2}y^2 + V^{\mu,\nu,\kappa}(x,y)$, where $V^{\mu,\nu,\kappa}(x,y) = x^3 - 3xy^2 + \overset{\mu}{a}(x^2+y^2) + \overset{\nu}{b}x + \overset{\kappa}{c}y$

(4) <u>Description</u> of the hyperbolic umbilic catastrophe

$\mathcal{C}$: catastrophe set, given parametrically by the equations

$$\begin{cases} \mu^2 = y^2 - x^2 \\ v = -2\mu x - 2xy \\ k = 2\mu y - y^2 - x^2 \end{cases}$$
, where x, y : parameters



NOTEBOOK

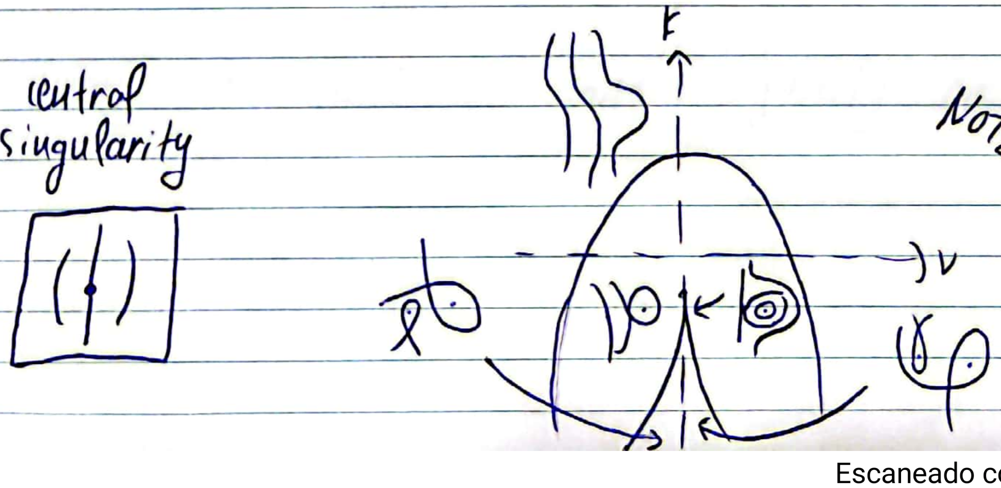
<u>Remarks</u>

(i) Cone-like structure, but now with a base consisting of the disjoint union of a smooth curve and a cusp-line.

(ii) The role of the parameter μ is the same as before.

<u>Bifurcation</u> diagram of the hyperbolic umbilic, $\mu \neq 0$ fixed

central singularity



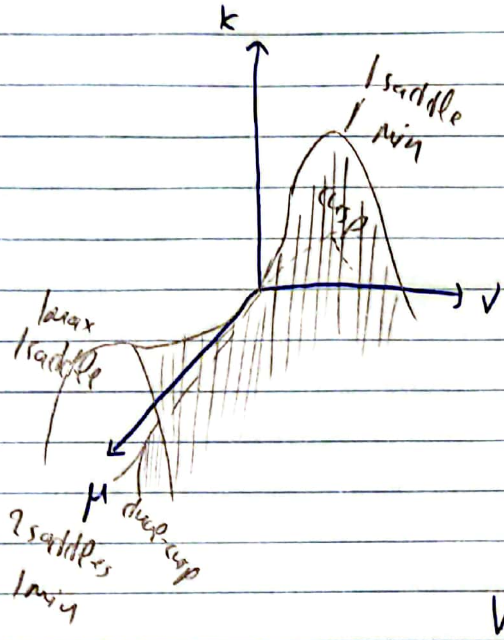
NOTEBOOK

Comments

We show the phase portraits related to the section $\mu=1$, as well as the central umbilic singularity.

- All kinds of subordinate bifurcations occur:
  - apart from dual cusp catastrophes, we also have the "ordinary" cusp catastrophe with normal form

$$H^{\mu,\nu}(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 + \frac{1}{2}\mu x^2 + \nu x$$



$$H^{\mu,\nu,k}(x,y) = \frac{1}{2}y^2 + V^{\mu,\nu,k}(x,y),$$

$$V^{\mu,\nu,k}(x,y) = x^3 + y^3 + \mu xy + \nu x + ky$$

Exercises

(1) Take the two hamilton. and justify the bifurc. diagram

Hints
- "start" with $\nu = k = 0$
- use aelfran

(2) Draw the rest of them