Orbivariant K-theory

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Orbispaces are spaces with extra structure. The main examples come from topological group actions $X \circlearrowleft G$ and are denoted $[X/G]$, their underlying space, or *coarse moduli space* being X/G. By definition, every orbispace is locally of the form $[X/G]$, but the group G might vary.

We shall work with orbispaces whose coarse moduli spaces are CW-complexes, and whose stabilizer groups are compact Lie groups. We also require the stratification of the coarse moduli space by the type of stabilizer group to be compactible with the CW structure. A convenient model for an orbispace is then given by a *topological groupoid* (see [2], [3]).

An orbispace always comes with a map to its coarse moduli space. By a *suborbispace* $\mathfrak{X}' \subset \mathfrak{X}$, we shall mean an orbispace obtained by pulling back along a subspace of the coarse moduli space.

If $\mathfrak{X}$ is an orbispace modeled by a topological groupoid $\mathcal{G}$, then a vector bundle over $\mathfrak{X}$ is a vector bundle over the space of objects of $\mathcal{G}$ equipped with an action of the arrows of $\mathcal{G}$. It is tempting to define K-theory as the Grothendieck group of vector bundles. But, as shown in [1], this is not always a good idea, even if $\mathfrak{X}$ is compact. For example, one needs the following condition to prove excision:

Definition 1 (Lück, Oliver [1]) *An orbispace $\mathfrak{X}$ has enough vector bundles if for every suborbispace $\mathfrak{X}' \subset \mathfrak{X}$ and every finite dimensional vector bundle V on $\mathfrak{X}'$, there exists a finite dimensional vector bundle W on $\mathfrak{X}$ and a linear embedding $V \hookrightarrow W$.*

This condition is not always satisfied (see Example 3, and section 5 of [1]).

Theorem 2 *Let $\mathfrak{X}$ be a compact orbispace (i.e. it's coarse moduli space is compact). Then the following are equivalent:*

1. *$\mathfrak{X}$ is a global quotient by a compact Lie group i.e. $\mathfrak{X} = [X/G]$ for some compact Lie group G acting on a compact space X.*
2. *$\mathfrak{X}$ has enough vector bundles.*
3. *There exists a vector bundle W on $\mathfrak{X}$ such that for every point x the action of $\mathrm{Aut}(x)$ on W_x is faithful.*

Proof. 1. $\Rightarrow$ 2. Let $X \circlearrowleft G$ be such that $\mathfrak{X} \simeq [X/G]$, and let $X' \subset X$ be the G-invariant subspace corresponding to $\mathfrak{X}' \subset \mathfrak{X}$. Let V be a vector bundle on $\mathfrak{X}'$ and let $\tilde{V}$ be the corresponding G-equivariant vector bundle on X'. It is well known that any equivariant vector bundle $\tilde{V}$ on a compact space X' embeds in one of the form $X' \times M$, where M is a representation of G. Let W be the vector bundle on $\mathfrak{X}$ corresponding to $X \times M \to X$. Since $\tilde{V}$ embeds in $X \times M$, the bundle V embeds in W.

2. $\Rightarrow$ 3. Suppose that $\mathfrak{X}$ has enough vector bundles, and let $\{U_i\}$ be a finite cover of $\mathfrak{X}$ such that $U_i \simeq [X_i/G_i]$. Let M_i be faithful representations of G_i, and let V_i be the vector bundles on U_i corresponding to $X_i \times M_i \to X_i$. Since M_i is faithful, the stabilizer groups act faithfully on the fibers of V_i. Let W_i be vector bundles on $\mathfrak{X}$ such that $V_i \hookrightarrow W_i$, and let $W := \bigoplus W_i$. Clearly, the stabilizer groups act faithfully on the fibers of W.

3. $\Rightarrow$ 1. Let P be the total space of the frame bundle of W. The stabilizer groups act faithfully on the fibers of W, hence they act freely on the fibers of P. Having no stabilizer groups, P is a space. We have $\mathfrak{X} = [P/O(n)]$ and so $\mathfrak{X}$ is a global quotient. □

Example 3 Let $P \to S^3$ be the principal BS^1-bundle classified by

$$1 \in [S^3, B(BS^1)] = \mathbb{Z}.$$

Then $\mathfrak{X} := [P/ES^1]$ does not have enough vector bundles.

Proof. We show that $\mathfrak{X}$ is not a global quotient by a compact Lie group. Indeed, suppose that $\mathfrak{X} = [X/G]$. Since $\mathfrak{X} \to S^3$ is homotopically non-trivial, the map $X \to S^3$ needs to be be a non-trivial G-fiber bundle with fiber $S^1\backslash G \circlearrowleft G$. Let $H = \mathrm{Aut}_G(S^1\backslash G)$ be the structure group of that bundle. All compact Lie groups have trivial π_2, therefore $[S^3, BH] = \pi_3 BH = \pi_2 H = 0$. The bundle $X \to S^3$ is trivial, a contradiction. □

More generally, any S^1-gerbe whose class in H^3 is non-torsion is an orbispaces without enough vector bundles.

Since there exist orbispaces without enough vector bundles, vector-bundle-K-theory is not a cohomology theory. So we need another definition for K-theory of orbispaces. Our preferred one, inspired by [4], is the following:

Definition 4 *Let $\underline{\mathbb{C}} := \mathbb{C} \times \mathfrak{X}$ be the trivial bundle. A cocycle for $K^0(\mathfrak{X})$ is a chain complex of $\underline{\mathbb{C}}$-modules (not necessarily locally constant) which is locally quasi-isomorphic to a bounded complex of finite dimensional vector bundles. Two K^0-cocycles on $\mathfrak{X}$ represent the same element in $K^0(\mathfrak{X})$ if they extend to a K^0-cocycle on $\mathfrak{X} \times [0, 1]$.*

If $\mathfrak{X}$ is a space (by which we mean that $\mathfrak{X}$ is a CW-complex, not necessarily compact) this definition recovers the usual topological K-theory of $\mathfrak{X}$.

References

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[4] Robert W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. *The Grothendieck Festschrift, Vol. III,* Progr. Math., 88, Birkhäuser, Boston, MA, pages 247–435, 1990.