Note on Wigner's Theorem on Symmetry Operations

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Wigner’s theorem states that a symmetry operation of a quantum system is induced by a unitary or an anti-unitary transformation. This note presents a detailed proof which closely follows Wigner’s original exposition.

INTRODUCTION

The states of a quantum system S are described by unit vectors f (i.e., vectors of norm 1) in some Hilbert space \( \mathcal{H} \). Assume that, conversely, to every unit vector \( h \) in \( \mathcal{H} \) corresponds a state of \( S \). This correspondence is, of course, not one-to-one since \( f \) and \( \lambda f \) describe the same state if \( \lambda \) is a scalar factor of modulus 1. The states of \( S \) are therefore a statement about unit rays.

Every vector \( f \) contained in the ray \( f (f \in \mathbb{R}) \) will be called a representative of \( f \). The transition probability from a state \( f \) to a state \( g \) equals \( |(f, g)|^2 \) where \( f, g \) are representatives of the rays \( f, g \), respectively. This suggests the introduction of the inner product of two rays by the definition

\[
(f, g) \equiv |(f, g)|, \quad (f, f) = 1,
\]

which is evidently independent of the choice of the representatives \( f, g \).

A symmetry operation \( T \) of the system \( S \) maps the states in \( \mathcal{H} \) either onto themselves or onto the states in some other Hilbert space \( \mathcal{H'} \), with preservation of transition probabilities. The second alternative corresponds to the mapping of one coherent subspace onto another. (See footnote 2.) In terms of rays, \( T \) defines a mapping, \( T' = T f \), of unit rays onto unit rays such that \( f'_1 \cdot f'_2 = f_1 \cdot f_2 \) if \( f'_1 = T f_1 \).

It has been shown by Wigner\(^3\) that every such ray mapping \( T \) may be replaced by a vector mapping \( U \) of \( \mathcal{H} \) onto \( \mathcal{H'} \) which is either unitary or anti-unitary.\(^4\) (For a precise formulation see Sec. 1.3 below.) For a long time this theorem has played a fundamental role in the analysis of symmetry properties of quantum systems.

The reason for returning to this question is the following. In Wigner’s book the theorem is not proved in full detail. The construction of the mapping \( U \), however, is clearly indicated, so that it is not difficult to close the gaps in the proof. In recent years several papers have appeared in which a proof of Wigner’s theorem is presented.\(^5,6\) To this writer most of these proofs seem unsatisfactory in one significant aspect: They obscure the quite elementary nature of Wigner’s theorem.\(^7\)

In addition, some authors state—or imply—the view that it is desirable, if not necessary, to depart from Wigner’s construction in order to arrive at a simple or rigorous demonstration of his theorem. This writer, on the contrary, has always felt that Wigner’s construction provides an excellent basis for an elementary and straightforward proof.

The present note is expository and contains no new results. It gives a complete proof of Wigner’s theorem, by a method which closely adheres to his original construction. The only change of any consequence is the following. While Wigner relates \( U \) to an orthonormal set defined once for all, the proof below uses orthonormal sets adjusted to the vectors under consideration. As a result, it suffices to employ sets of at most two or three vectors.

Remarks on the notation. Re \( \lambda \) and Im \( \lambda \) denote, respectively, the real and the imaginary part of the complex number \( \lambda \), and \( \lambda^* \) its complex conjugate.

\(^1\) Here \( f \) corresponds of course to a wavefunction \( \psi \).
\(^2\) In this note vectors will be denoted by italics and scalars by lower-case Greek letters. The product of a vector \( f \) by the scalar \( \lambda \) will be written \( f_\lambda \).
\(^3\) If superselection rules hold for \( S \), \( \mathcal{H} \) will be considered a coherent subspace of the Hilbert space of all states. See the discussion in Wightman [A. S. Wightman, Nuovo Cimento Suppl. 14 (1959), p. 81].

\(^5\) Although Wigner did not explicitly formulate his theorem in terms of rays, it is essentially equivalent to the one stated above and certainly follows from the theorem proved below.\(^3\)
\(^6\) For a bibliography and a critical analysis of the proofs, see Uhlhorn, Ref. 6. The recent paper by Lomont and Mendelson [J. S. Lomont and P. Mendelson, Ann. Math. 78, 548 (1963)] should be added to his list.

This criticism does not apply at all to the very interesting papers by Emch and Piron (G. Emch and C. Piron, J. Math. Phys. 4, 469 (1963)) and by Uhlhorn\(^4\), who start from more general premises and, consequently, obtain more comprehensive results.
1. STATEMENT OF THE THEOREM

1.1. Preliminary remarks on rays. Let \( \mathcal{H} \) be a complex Hilbert space—which may be finite dimensional—with vectors \( f, g, \ldots \). The inner product \( (f, g) \) of two vectors \( f, g \) has Hermitian symmetry, i.e., \((g, f) = (f, g)^*\), and for any complex scalar \( \lambda \)

\[
(f, g\lambda) = (f, g)\lambda.
\]  

(1)

\( ||f|| = (f, f)^\frac{1}{2} \) is the norm of \( f \).

A ray \( f \) in \( \mathcal{H} \) is the set of all vectors \( f\sigma \), where \( f_0 \) is a fixed vector in \( \mathcal{H} \) and \( \sigma \) any scalar of modulus 1. Every vector \( f \in \mathcal{H} \) is an element or a “representative” of \( f \). Two vectors \( f' \), \( f'' \) are equivalent if they belong to the same ray, which is the case if and only if \( f'' = f\omega \) (\( |\omega| = 1 \)). It is clear that a ray \( f \) is uniquely determined by any one of its representatives \( f \), and we write

\[
f = [f].
\]  

(1a)

\( 0 \) is the ray consisting of the vector 0.

The inner product of two rays \( f, g \) is defined by

\[
f \cdot g = \|f\| \|g\| (f, g)\]

(1b)

and the norm of the ray \( f \) by

\[
|f| = (f, f)^\frac{1}{2} = \|f\| (f \in f).
\]  

(1c)

A unit ray is a ray of norm 1.

For nonnegative real scalars \( \rho \) we define

\[
f_\rho = [f\rho] \quad (f \in f),
\]  

(2)

i.e., if \( f_\rho \in f \), the elements \( g \) of \( f_\rho \) are given by \( g = f_\rho \sigma \) (\( |\rho| = \rho \)). Clearly,

\[
(f_\rho)\sigma = f(\rho\sigma), \quad |f_\rho| = |f| \rho,
\]  

(2a)

\[
f_\rho \cdot g = (f, g)^\frac{1}{2} \rho|g|.
\]  

(2b)

Every ray \( a \) may be expressed in the form

\[
a = e^\rho \quad (|e| = 1, \rho \geq 0).
\]  

(2c)

In all cases \( \rho = |a| \). If \( a = 0 \), then \( \rho = 0 \), and the unit ray \( e \) may be chosen arbitrarily. If \( a \neq 0 \), \( e \) is uniquely determined as \( a^{-1} \).

1.2. It is reasonable to impose the following conditions on a symmetry operation \( T \).

(a) \( T \) is defined for every unit ray \( e \) in \( \mathcal{H} \), and \( e' = Te \) is a unit ray in \( \mathcal{H} \).

(b) \( Te_1 = Te_2 \) (preservation of transition probabilities).

(c) If \( Te_1 = Te_2 \), then \( e_1 = e_2 \) (the mapping is one-to-one).

(d) Every unit ray \( e' \) in \( \mathcal{H}' \) is the image of some \( e \) in \( \mathcal{H} \) (the mapping is onto the unit rays in \( \mathcal{H}' \)). It is easily seen that (c) is superfluous, because it is an immediate consequence of (b). By Schwarz’s inequality, two unit rays \( f, g \) coincide if and only if \( f \cdot g = 1 \). Hence if \( Te_1 = Te_2 \), then \( e_1 = e_2 \) by (b).

In order to make the structure of the theorem more transparent we also drop the condition (d) and reinstate it in a corollary.

1.3. Thus our aim is the proof of the following

**Main Theorem.** Let \( e' = Te \) be a mapping of the unit rays \( e \) of a Hilbert space \( \mathcal{H} \) into the unit rays \( e' \) of a Hilbert space \( \mathcal{H}' \) which preserves inner products, i.e., such that

\[
Te_1 \cdot Te_2 = e_1 \cdot e_2.
\]  

(3)

Then there exists a mapping \( a' = Ua \) of all vectors \( a \) in \( \mathcal{H} \) into the vectors \( a' \) in \( \mathcal{H}' \) such that

\[
Ua \in Ta \quad \text{if} \quad a \in a
\]  

(4)

(if \( Ta \) is defined) and, in addition

\[
\begin{align*}
(a) \quad U(a + b) &= Ua + Ub, \\
(b) \quad U(a\lambda) &= (Ua)x(\lambda), \\
(c) \quad (Ua, Ub) &= x(a, b),
\end{align*}
\]  

(5)

where either \( x(\lambda) = \lambda \) for all \( \lambda \) or \( x(\lambda) = \lambda^* \) for all \( \lambda \).

A vector mapping \( U \) which satisfies (4) is called compatible with \( T \).

\( U \) is isometric since \( ||Ua|| = ||a|| \) by (5). It is a linear or an antilinear isometry according as \( x(\lambda) = \lambda \) or \( x(\lambda) = \lambda^* \).

**Corollary.** If the \( T \) of the theorem is a mapping onto all unit rays in \( \mathcal{H} \) in which case we call it a “ray correspondence”—\( U \) is a mapping onto \( \mathcal{H}' \). It is unitary if \( x(\lambda) = \lambda \) and anti-unitary if \( x(\lambda) = \lambda^* \).

The corollary is an immediate consequence of the theorem.

1.4. The one-dimensional case is of course trivial. \( \mathcal{H} \) contains only one unit ray \( e \), and \( T \) is completely determined by \( Te = e' \). Let \( e \in e \), and \( e' \in e' \).

The two vector mappings \( U_1(ea) = e' \alpha \) and \( U_2(ea) = e' \alpha^* \) are compatible with \( T \). The first is linear, the second antilinear.

Hereafter we assume \( \mathcal{H} \) to be at least two dimensional.

1.5. It is worth mentioning that, if \( \dim \mathcal{H} \geq 2 \), the linear or antilinear character of \( U \) may be
expressed in terms of $T$. It describes, therefore, an \textit{intrinsic} property of the mapping $T$ and is independent of the choice of $U$.

Consider three rays $a_1$, and let $a_i \in a_j$. The expression

$$\Delta(a_1, a_2, a_3) = (a_1, a_2)(a_2, a_3)(a_3, a_1)$$

is \textit{independent} of the choice of the representatives $a_i$ and is therefore a function of the rays $a_i$. In fact, if $a_i$ are replaced by $a'_i = a_i\tau_i$ ($|\tau_i| = 1$) the factors $\tau_i$ cancel in $\Delta$. It follows now from (5c) that

$$\Delta(Te_1, Te_2, Te_3) = \chi(\Delta(e_1, e_2, e_3)).$$

As it should be, this criterion is vacuous if $\dim \mathcal{C} = 1$ because then $e_i = e$, and $\Delta = 1$. But if $\dim \mathcal{C} \geq 2$, $\Delta$ is not always real and may serve to distinguish linear from antilinear mappings. [Let $\Delta = \epsilon_1^2$.]

Because then $e_i = e$, and $\Delta$ is compatible with) a uniquely defined ray mapping $T$ by the equation

$$T[a] = \{ua\}$$

[see (1a)]. It is clear that every (linear or antilinear) isometry—in view of (5b) and (5c)—induces a ray mapping $T$ which preserves inner products. Wigner's theorem asserts that no other ray mappings of this kind exist.

2. EXTENSION OF THE MAPPING $T$

Before constructing $U$ we extend, following Wigner, $T$ from a mapping of unit rays to a mapping of all rays $a$ in $\mathcal{C}$ into the rays $a'$ in $\mathcal{C}'$ by defining

$$T(\rho) = (Te)\rho \quad (\rho \geq 0, |e| = 1).$$

Note that (6) defines $T$ unambiguously for every ray $a = e\rho$. If $a = 0$, then $\rho = 0$, hence $T0 = 0$. If $a \neq 0$, both $\rho$ and $e$ are uniquely determined [see (2c)].

For the extended mapping we have

\begin{align*}
(a) & \quad T(\rho) = (Te)\rho \quad (\rho \geq 0), \\
(b) & \quad T(a_1, a_2) = a_1 \cdot a_2, \quad (c) \quad |Ta| = |a|.
\end{align*}

(7)

[(a) if $a = e\rho$, both sides of (7a) equal $(Te)\rho e$; (b) Set $a_1 = e\rho_1$. The assertion follows from (2b) and (3), and if $a_1 = a_2 = a$, we obtain (c).]

In the sequel we deal throughout with the \textit{extended} mapping $T$—which is assumed to be given—and construct $U$ so as to be compatible with it. Equation (4) is the \textit{only} condition imposed on $U$. [(5) results from the construction.]

For later use we note the following. If, in the course of the construction, $U$ has been defined [in accordance with (4)] for all multiples $a\lambda$ of a vector $a \neq 0$, then by (7)

$$||ua|| = ||a||, \quad U(a\lambda) = (Ua)x_\lambda(\lambda),$$

$$x_\lambda(1) = 1, \quad |x_\lambda(\lambda)| = |\lambda|,$$

where $x_\lambda(\lambda)$ is a uniquely defined function of $a$ and $\lambda$. If $U$ has been defined for $a$ and $b$,

$$||(Ua, Ub)|| = ||(a, b)||.$$}

In Sec. 3 the mapping $U$ is constructed for a subclass of all vectors. The partially defined $U$ is analyzed in Sec. 4, and in Sec. 5 the construction of $U$—and the proof of the main theorem—is completed.

3. PARTIAL CONSTRUCTION OF $U$

3.1. Preliminary remarks. Let $f_\rho, (\rho = 1, \ldots, m)$ be $m$ orthonormal rays (m finite!) so that $f_\rho f_\sigma = \delta_{\rho\sigma},$ and set $f_\rho^* = T\rho$. If $f_\rho \in f_\rho'$, then $f_\rho f_\rho' = \delta_{\rho\rho'}.$ Let $a = \sum f_\rho a_\rho$, and $a = \{a\}$. For any $a' \in Ta$,

$$a' = \sum f_\rho a'_\rho, \quad |a'_\rho| = |a_\rho|.$$ (9)

Proof: Note that $|a'|^2 = |a|^2, \quad |f_\rho'a'| = |f_\rho a|,$ and $(f_\rho, a) = \alpha_\rho.$ Therefore

$$||a' - \sum f_\rho(f_\rho', a')(a')^*||^2 = ||a'||^2 - \sum |f_\rho(f_\rho', a')|^2$$

$$= ||a||^2 - \sum |f_\rho(a)|^2 = ||a - \sum f_\rho(f_\rho, a)||^2 = 0,$$

hence $a' = \sum f_\rho'(f_\rho', a'),$ and the assertion follows, with $a'_\rho = (f_\rho', a').$

3.2. Fix a unit ray $e$ in $\mathcal{C}$, and let $e' = Te$, so that $|e'| = 1$. Select $e \in e$, and $e' \in e'$. We define $A$.

$$Ue = e'$$

in accordance with (4).\textsuperscript{11}

Denote by $\mathcal{O}$ the set of vectors in $\mathcal{C}$ orthogonal to $e$, by $e'$ the set of vectors in $\mathcal{C}'$ orthogonal to $e'$. Every vector $a$ in $\mathcal{C}$ has a unique decomposition

$$a = ea + z \quad (z \in \mathcal{O});$$

viz., $a = (e, a), z = a - e(e, a)$. In this section we construct $U$ for those $a$ for which $\alpha = 0$ or 1.

Let $a = e + z (e \in \mathcal{O}, z \neq 0)$, and set $f = z/||z||$.

\textsuperscript{11} The selection of $e'$ constitutes the only arbitrary choice in the construction of $U$. All other definitions are uniquely determined by $e'$.\textsuperscript{11}
Let a, f, z be the corresponding rays, so that
\[ z = f \ ||z||, \quad |f| = 1. \]

If \( a' \in T_a \), and \( f' \in T_f \), then, by (9), \( a' = e'a' + f'a', \ |a'| = 1, |a| = ||z||. \) Hence \( T_a \) contains a uniquely determined vector \( a'' \) of the form \( e' + f'b' \, (|b'| = ||z||) \). Setting \( f'b' = Vz \) we define
\[ V(z) = e' + Vz \quad (z \in \mathcal{O}, \ Vz \in \mathcal{O}'). \]

Clearly, \( Vz = f'b' \in \langle T_f \rangle ||z|| = Tz \), and we are allowed to set
\[ \text{C.} \quad Uz = Vz(U(e + z) - U) \quad (z \in \mathcal{O}). \]

If \( z = 0 \), we set \( Vz = 0 \), so that \( B \) reduces to \( A \).

In the next section we analyze the mapping \( V \) of \( \mathcal{O} \) into \( \mathcal{O}' \) in greater detail.

4. ANALYSIS OF THE MAPPING \( V \)

4.1. The real part of \( (Vw, Vz) \). Let \( w, z \) be in \( \mathcal{O} \). By C, B, and (8b),
\[ \Re (Vw, Vz) = (w, x), \quad (Vw, Vz) = (w, x) \quad \text{real.} \quad (12) \]

4.2. It will now be shown—in the remainder of this section—that, for any two nonvanishing \( y, z \) in \( \mathcal{O} \), (a) \( V(y + z) = Vy + Vz \), (b) \( X_\lambda(y) = X_\lambda(z) \) [see (8)], (c) \( (Vy, Vz) = X_\lambda(y, z) \).

4.3 Set \( f_1 = y/||y|| \). If \( \text{dim} \mathcal{O} \geq 2 \) all vectors in \( \mathcal{O} \) are multiples of \( f_1 \), hence
\[ y = f_1, \quad z = f_1. \quad (13a) \]

If \( \text{dim} \mathcal{O} \geq 3 \) choose a second unit vector \( f_2 \) in \( \mathcal{O} \) orthogonal to \( f_1 \) (whether or not \( y \) and \( z \) are independent) such that
\[ y = f_1, \quad z = f_1 + f_2. \quad (13b) \]

In both cases let \( L \) be the set of linear combinations of the \( m \) orthonormal vectors \( f_1, f_2, \ldots, f_m \) (\( m = 1 \) or \( m = 2 \)).

4.4. The functions \( X_\lambda(a) \). Set \( f_1 = \{f_1\} \). Then \( f' = Vf_1 \in T_{f_1} \), and the vectors \( f'_1 \) are orthonormal. By (8),
\[ V(f_1) = f'_1X_\lambda(a), \quad |X_\lambda(a)| = |a|. \quad (14) \]

Applying (12) to \( f_1 \) and \( f_2 \), we obtain
\[ \Re (X_\lambda(a)X_\lambda(b)) = \Re (a^*b). \quad (15) \]

Set \( \alpha = 1 \). Since \( X_\lambda(1) = 1 \) we conclude from (12) and (12a) that
\[ \Re (X_\lambda(b)) = \Re (b) \quad \text{for real } b. \quad (15a) \]

4.5. Let \( x = \sum f_\alpha \). By (9), \( Vz = \sum f'\alpha \), \( |\alpha'| = |\alpha| \). We prove first that \( \alpha'_2 = X_\lambda(\alpha) \). This is trivial if \( \alpha = 0 \). If \( \alpha \neq 0 \), set \( \gamma = \alpha^{*-1} \). Then
\[ \sum f_\gamma X_\lambda(\alpha) = \sum f_\gamma X_\lambda(1) = 1, \ i.e., \alpha'_2 = X_\lambda(\alpha). \quad (15b) \]

We show next that \( X_\lambda(\alpha) = \chi_\lambda(\alpha) \) if \( m = 2 \). Let \( w = \sum f_\alpha \). Then \( Vw = \sum f'_\alpha \), and \( V(wa) = \sum f'_\alpha X_\lambda(a), \) by (8). Thus \( \chi_\lambda(\alpha) = X_\lambda(\alpha) = \chi_\lambda(a) \). As a result,
\[ V(\sum f_\alpha) = \sum f'_\alpha X_\lambda(a). \quad (16) \]

4.6. Determination of \( \chi_\lambda(b) \). (1) Set \( \beta = i \). Then \( |\chi_\lambda(i)| = 1, \ Re \chi_\lambda(i) = 0 \), thus \( \chi_\lambda(i) = \eta i, \eta = 1 \) or \( -1 \). (2) For any complex \( \gamma \), \( \eta \) = \( \Re (\gamma^* \eta) \).

Hence, from (15), \( \eta \Re \chi_\lambda(i)X_\lambda(\beta) \) = \( \eta \Re (\chi_\lambda(i)X_\lambda(\beta)) = \eta \Re (\gamma^* \eta) \eta \Re \beta \). Combining this with (15a),
\[ \chi_\lambda(\beta) = \beta \text{ if } \eta = 1, \chi_\lambda(\beta) = \beta^* \text{ if } \eta = -1. \quad (17) \]

Note the obvious relations:
\[ \begin{align*}
(a) & \quad \chi_\lambda(\alpha + \beta) = \chi_\lambda(\alpha) + \chi_\lambda(\beta), \\
(b) & \quad \chi_\lambda(\alpha \beta) = \chi_\lambda(\alpha)\chi_\lambda(\beta), \\
(c) & \quad \chi_\lambda(\alpha^*) = \chi_\lambda(\alpha^*). 
\end{align*} \]

4.7. The structure of \( V \). Let \( w = \sum f_\alpha \) and \( x = \sum f_\beta \), be two vectors in \( L \). From (16) and from the properties of \( \chi \), just stated we draw the following conclusions. By (a), \( V(w + z) = Vw + Vz \), by (b), \( V(\lambda x) = (Vx)\chi_\lambda(\lambda) \). Since both \( y \) and \( z \) belong to \( L \) this proves the assertions (a) and (b) of Sec. 4.2, with \( \chi_\lambda(\lambda) = \chi_\lambda(\lambda) \). To establish (c) in (4.2) we note that \( y, z = \rho^* \sigma \) by (13b), and (c) \( (Vy, Vz) = \chi_\lambda(\rho^* \chi_\lambda(\sigma) = \chi_\lambda(\rho^* \chi_\lambda(\sigma), \) Q.E.D.

By (b) in Sec. 4.2, \( \chi_\lambda(\lambda) \) is the same function, say, \( \chi(\lambda) \), for every nonvanishing vector \( \sigma \) in \( \mathcal{O} \). To sum up, the mapping \( V \) has the following properties:
\[ \begin{align*}
(a) & \quad V(y + z) = Vy + Vz, \\
(b) & \quad V(\lambda x) = (Vx)\chi_\lambda(\lambda), \\
(c) & \quad (Vy, Vz) = \chi((y, z)).
\end{align*} \]

\(^{13}\) \( e \) and \( f \) are orthonormal, so are \( e' \) and \( f' \). The definitions A and B are the crucial steps in Wigner’s construction.

\(^{14}\) If \( m = 1 \), Sec. 4.5 may be omitted since Eq. (16) reduces then to Eq. (14).
where $\chi$ is one of the functions in (17). [The equations (18) have been explicitly proved for nonvanishing $y$ and $z$. But they hold trivially if $y = 0$ or $z = 0$.]

5. THE CONSTRUCTION OF $U$ COMPLETED

It remains to define $U$ for vectors $a = ax + z \xi (z \in \xi^a)$ for which $\alpha \neq 0, 1$. [See (10).] Set $b = ax + z\xi$, so that $a = bx$, and $T_a = (Tb) |a|$ if $a, b$ are the corresponding rays. $U_b (\xi T_b)$ is defined by $B$ in Sec. 3. Hence $(Ub)x(\xi) \in T_b$, and we may therefore define $Ua = (e' + V(za^{-1}))x(\alpha)$, or, by (18),

$$U(ax + z) = e'(x(a) + Vz \xi \xi) (z \in \xi^a).$$

If $\alpha = 1$ or $0$, $D$ coincides with $A, B$, or $C$ of Sec. 3. Thus it defines the mapping $U$ for all vectors $a$ in $3C$.

By virtue of (18) it is an immediate consequence of $D$ that $U$ satisfies all conditions (5) of the theorem.

For an example, let $a_i = ax_i + z_i \xi (j = 1, 2)$. Then $(a_1, a_2) = ax_1^2 + z_1 \xi + z_2 \xi$, and $(Ua, Ua) = x(ax_1) + x(z_1 \xi) + x(z_2 \xi) = x(ax_1) + x(z_1 \xi) + x(z_2 \xi).

This concludes the proof of the main theorem.

6. UNIQUENESS OF $U$

It is of course important to know to what extent $U$ is determined by a ray mapping $T$ which preserves inner products. Without using the main theorem in this section the following can be asserted. ($T$ stands for the extended mapping)

(a) Let $U$ be a vector mapping compatible with $T$. If $a_1, a_2$ are (linearly) independent, so are $Ua_1, Ua_2$.

Proof: Two vectors $a_i$ are independent if and only if $G(a_1, a_2) = (a_1, a_1)(a_2, a_2) - |(a_1, a_2)|^2 > 0$. Since, by (8b), $G$ is not changed by the mapping $U$, the assertion follows.

(b) If $U_2$ and $U_1$ are compatible with the same $T$, then $U_2a = U_1a$ for every $a \neq 0$.

If $\tau(a) = \theta$ (independent of $a$), we write $U_2 = U_1\theta$. A mapping $U$ is additive if $U(a + b) = Ua + Ub$.

Theorem 2. If two additive vector mappings $U_2$ and $U_1$ are compatible with the same $T$, and dim $3C \geq 2$, then $U_2 = U_1\theta$. [ $|\theta| = 1.$]

Proof: We proceed in two steps. (1) If $a, b$ are independent, $\tau(a) = \tau(b)$. Set $c = a + b$. Then $U_c = U_a + U_b. U_c = U_a + U_b$, and $U_c = (U_c)a + (U_c)b$. Therefore

$$(U_a)a \tau(a) + (U_b)b \tau(b) = (U_a + U_b)\tau(c).$$

Since $U_a$ and $U_b$ are independent, $\tau(c) = \tau(a) = \tau(b)$.

(2) Fix a vector $a_0 \neq 0$ in $3C$, and set $\tau(a_0) = \theta$. For every vector $a \neq 0, \tau(a) = \theta$. If $a$ and $a_0$ are independent, this follows from (1). If $a = a_0 \mu (\mu \neq 0)$, choose $b$ independent of $a_0$ (and hence of $a$). Then, by (1), $\tau(b) = \tau(a) = \theta$.

Let $U_2 = U_1\theta$. If $U_1(a) = (U_1(a)x(\alpha), then also $U_2(a) = (U_2(a)x(\alpha)$.

In fact, $U_2(a) = (U_2(a)x(\alpha), \theta = (U_1(\alpha)x(\alpha), \theta$ (19) in accordance with our result in Sec. 1.5.

APPENDIX. WIGNER'S THEOREM IN QUATERNION QUANTUM THEORY

In recent years there has been some interest in a modification of the quantum theoretical formalism which consists in replacing the complex Hilbert space of quantum states by a quaternion Hilbert space.

We wish to indicate the changes in the theorem and in its proof that must be made. The above exposition is so arranged that these changes are concentrated in a few places.

1. Preliminary remarks on quaternions. Let $Q$ be the set of all quaternions. We write a quaternion $\lambda$ in the form $\lambda = e_0 + e_1 i_1 + e_2 i_2 + e_3 i_3$. Here, $\lambda$, are real numbers, $i_1 = 1$ (i.e., $i_1 \lambda = \lambda i_1 = \lambda$ for every $\lambda \in Q$), while $i_1 (r = 1, 2, 3)$ are the imaginary units, with the multiplication rules

$$i_1^2 = -1, \quad i_1 i_2 = -i_2 i_1 = i_3$$

where $s, t$ is an integral of $1, 2, 3$.

The conjugate $\lambda^\ast$ of $\lambda$ is defined by

$$\lambda^\ast = \lambda_0 \lambda_1 - \sum_{r=1}^3 \lambda_r i_r$$

so that $(\lambda^\ast)^\ast = \lambda$. A quaternion is real if $\lambda^\ast = \lambda$, i.e., $\lambda_0 = \lambda_0 = 1$.

The real quaternions, and only they, commute with all of $Q$. In general we denote the real part of $\lambda$ by

$$Re \lambda = \frac{1}{2}(\lambda + \lambda^\ast) = \lambda_0$$

Note that $Re \lambda^\ast = Re \lambda$. Let $K = \sum_1^3 \kappa_1 i_1$. Since $(\kappa i_1)^\ast = i_1^2 i_1^2 = i_1^2 (\text{for all } \mu, \nu), (\kappa^\ast)^\ast = \lambda^\ast = \kappa^\ast$.

For all $\mu, \nu$,

$$Re (i_1^2 i_1^2) = Re (i_1^2 i_1^2) = \delta_{11}$$

It follows that

$$Re (\kappa^\ast) = Re (\kappa) = \sum_1^3 \kappa_1 i_1, \quad (A2a)$$

$$Re (\kappa) = Re (\kappa^\ast)$$

$\text{[A1]}$
In particular, $\lambda^*\lambda$ and $\lambda^*\lambda$ being real, 
\[ \lambda^*\lambda = \lambda^*\lambda = \sum \lambda^2 = |\lambda|^2, \]
where $|\lambda|$ is the modulus of $\lambda$. We have $|\lambda\lambda| = |\lambda| |\lambda|$ since $|\lambda\lambda|^2 = (\lambda^*\lambda)^* = (|\lambda|^2 |\lambda|^2) = |\lambda|^4$. If $\lambda \neq 0$, $\lambda^{-1} = |\lambda|^{-2} \lambda^*$, $\lambda^{-1} = |\lambda|^{-1} = 1$. 

$Q$ may be considered a four-dimensional real vector space. (A2a) defines then an inner product in $Q$, and by (A2) the units $i$ form an orthonormal basis. Hence 
\[ \lambda = \Re (i^*\lambda). \]  

More generally, let $j$, $(\nu = 0, 1, 2, 3)$ be four orthonormal quaternions, so that $\Re (j^*j) = \delta_{\nu}$. Then every $\kappa$ may be written as 
\[ \kappa = \sum \Re (j^*\lambda)j. \]  

For a later application we add a few words about the automorphism 
\[ \lambda' = \sigma_{\gamma}(\lambda) = \gamma^{\lambda} \gamma^{-1} = \gamma \lambda^* \]  
where $\gamma$ is a fixed quaternion of modulus 1. Clearly 

(a) $\sigma_{\gamma}(\kappa + \lambda) = \sigma_{\gamma}(\kappa) + \sigma_{\gamma}(\lambda)$, 
(b) $\sigma_{\gamma}(\kappa\lambda) = \sigma_{\gamma}(\kappa)\sigma_{\gamma}(\lambda)$, 
(c) $\sigma_{\gamma}(\lambda^*) = \sigma_{\gamma}(\lambda^*)^*$. 

In addition, 

(d) $\Re \sigma_{\gamma}(\lambda) = \Re \lambda$, 
(e) $\Re (\sigma_{\gamma}(\kappa)^*\sigma_{\gamma}(\lambda)) = \Re (\kappa^*\lambda)$. 

[(d) follows from $\Re (\gamma\lambda)^* = \Re (\gamma^*\gamma) = \Re \lambda$, and this in turn implies (e) because $\sigma_{\gamma}(\kappa)^*\sigma_{\gamma}(\lambda) = \sigma_{\gamma}(\kappa^*\lambda)$.] By (e), $\sigma_{\gamma}$ is an orthogonal transformation on $Q$. Conjugation is also an orthogonal mapping, by (A2a). Combining it with $\sigma_{\gamma}$ one obtains a second type of orthogonal transformation, 
\[ \lambda' = \sigma_{\gamma}(\lambda^*). \]  

2. Wigner's theorem. The states of the system $S$ are again put in a one-one correspondence with the unit rays in the Hilbert space $\mathfrak{C}$. The discussion in Sec. 1.1 remains unchanged except that the scalars $\lambda$ and the values of the inner products $(f, g)$—in $\mathfrak{C}$ and $\mathfrak{C}$—are now quaternions. More generally, the whole content of Secs. 1–6 remains applicable with the exception of those instances where (a) specific properties of complex numbers are used or (b) the commutative law of multiplication is applied. 

The only instance of type (a) is the determination of $\chi_1$ in Sec. 4.6 and, consequently, the characterization of $\chi(\lambda)$ in the statement of the main theorem. The only instance of Type (b) is Eq. (19) in Sec. 6.

These two points are now re-examined.

3. The two-dimensional case. In the quaternion case Wigner's theorem no longer holds if dim $\mathfrak{C} = 2$. Every vector $z$ in $\mathfrak{C}$ has now the form $f_1\alpha$ and 
\[ f_1(\alpha) = f_1\chi(\alpha) \]  
where (a) specific properties of complex numbers are used or (b) the commutative law of multiplication. 

Consider the three equations (15), (15a), and (15b). From (A4) and (A4a) we obtain two types of solutions, viz., 

(1) $\chi_1(\alpha) = \sigma_{\gamma}(\alpha)$, 
(2) $\chi_1(\alpha) = \gamma \sigma_{\gamma}(\alpha)^*$. 

(\text{It is not difficult to show that no other solutions exist.})

Now the proof of the main theorem in Sec. 5 is based, in part, on Eq. (18b), whose derivation in turn depends on the relation (b) at the end of Sec. 4.6, namely, $\chi_1(\alpha\beta) = \chi_1(\alpha)\chi_1(\beta)$. While the first solution in (A5) satisfies this relation we find for the second $\chi_1(\alpha\beta) = \gamma\sigma_{\gamma}(\alpha)^* = \gamma \sigma_{\gamma}(\beta)^*\sigma_{\gamma}(\alpha)^* = \chi_1(\beta)\chi_1(\alpha)$. The order of the factors is reversed; $\chi_1$ is an antiautomorphism.

Choose, for simplicity, $\gamma = 1$, so that $V(f_1\beta) = f_1\beta^\delta$. As the arguments of Sec. 5 show we may set 
\[ U(\alpha + \beta) = (e^\dagger + f_1(\alpha^{-1})^*\alpha \) (\alpha \neq 0). \]  

For convenience we define a new mapping $U_6$ compatible with $T$ by 
\[ U_6(\alpha + \beta) = (e^\dagger + f_1(\alpha^{-1})^*\alpha \) (\alpha \neq 0), \]  
\[ U_6(f_1\beta) = f_1\beta \]  
(\text{$U_6$ differs from $U$ only if $\alpha = 0$.})

To disprove Wigner's theorem in this case it remains to show (1) that $U_6$ actually induces a ray mapping $T$ with the required properties \{i.e., that no conditions have been overlooked that might rule out the second solution in (A5); (2) that no additive vector mapping is compatible with $T$. 

(1) By straightforward computation one verifies that, for every vector $a = \alpha + f_1\beta$, $U_6(\alpha\lambda) = (U_6\alpha)\lambda$, and that \[ [(U_6\alpha_1, U_6\alpha_2)] = [(\alpha_1, \alpha_2)]. \]  
Thus $U_6$ induces indeed a ray mapping $T$ which preserves inner products. 

(2) will be proved by contradiction. Let $W$ be an additive vector mapping compatible with $T$. Then 
\[ W(\alpha + \beta) = (U_6(\alpha + \beta))\Phi(\alpha, \beta) \]  
where $\Phi(\alpha, \beta) = 1$ if $(\alpha, \beta) \neq (0, 0)$. In particular, 
\[ W(f_1\beta) = f_1\beta^\delta \]  
and $\Phi(\alpha, \beta) = f_1\beta^\delta \Phi(\alpha, \beta)$. Setting $\eta(\alpha) = \alpha^\delta \Phi(\alpha, \alpha)$ and 
\footnote{It should be added that the remarks in Sec. 1.5 do not apply to the quaternion case. The proof that $\Delta$ does not depend on the choice of the representatives $a_i$ uses commutativity of multiplication.}
Assume \( \xi(\beta) = \beta \Phi(0, \beta) \) we conclude from the additivity of \( W \) that

\[
e' \eta(\alpha) + f_1^s \eta(\beta) = (e' + f_1^s (\alpha^* \beta^*) \alpha \Phi(\alpha, \beta)\).
\]

Assume \( \alpha \neq 0, \beta \neq 0 \). Then \( \eta, \xi \) and \( \Phi \neq 0 \), and

\[
\alpha^* \xi(\beta) = \beta^* \eta(\alpha).
\]

(6)

Setting, in succession, \( \alpha = \beta = 1, \beta = 1 \), and \( \alpha = 1 \), one finds \( \xi(1) = \eta(1), \eta(\alpha) = \alpha^* \eta(1), \xi(\beta) = \beta^* \eta(1) \). Multiplying (A6) with \( \eta(1)^{-1} \) from the right, we finally obtain \( \alpha^* \beta^* = \beta^* \alpha^* \) or \( \beta \alpha = \alpha \beta \), which is absurd.

3. Theorem of Uhlhorn. We first derive a further condition on \( \xi(\beta) \).

Let \( \beta = \xi(\beta) \). By (15), \( \Re (\beta \xi(\beta)) = \Re (\xi \beta) = \delta_{ss} \).

Let \( \beta = \sum_{i=1}^s \beta_i i \). By (15), \( \Re (\sum_{i=1}^s \beta_i i \xi(\beta)) = \Re (\xi(\beta)) = \beta \), [see (A3)], so that, by (A3a)

\[
\xi(\beta) = \sum_{i=1}^s \beta_i i \xi(\beta).
\]

(7)

By (15b), \( \xi(-1) = -1 \); hence by (A7), \( \xi(-\beta) = -\xi(\beta) \). Applying (A7) to \( j_i \) (\( r > 0 \)) we find

\[
\begin{align*}
 j_i 
&= \Re (j_i \xi(\beta)) = \Re (j_i \beta) = \delta_{ss} \text{ if } r > 0, \\
&= \xi(\beta) \text{ if } r = 0.
\end{align*}
\]

(8)

If \( i, j, k \) is an even permutation of \( (1, 2, 3) \). Together with \( j_0 = 1 \), this shows that the \( j_i \) satisfy the multiplication rules of the units \( i \).

It is well known—and easily proved—that then \( j_i = \gamma_i \gamma^{-1} = \sigma_k(i) \) for some fixed \( \gamma \) of modulus 1. Inserting this in (A8) we finally obtain

\[
\xi(\beta) = \sigma_k(\xi(\beta)).
\]

(9)

This solution satisfies all three relations listed at the end of Sec. 4.6, and the arguments in Sec. 4.7 and Sec. 5 apply without change. Thus the main theorem is valid, but \( \chi(\lambda) \) is an automorphism \( \sigma_k(\lambda) \), and \( U \) a semilinear isometry.

5. Theorem 2 of Sec. 6 holds. The transition from \( U_1 \) to \( U_2 = U \theta \), however, has now more radical consequences. Instead of Eq. (19) we have

\[
U_4(\lambda) = U_1(\lambda) \chi(\lambda) \theta = (U_1(\lambda) \theta \theta^{-1} x(\lambda) \theta).
\]

Assuming \( x(\lambda) = \sigma_k(\lambda) \),

\[
U_4(\lambda) = (U_2(\lambda) \chi' \lambda),
\]

(10)

\[
\chi'(\lambda) = \theta^{-1} x(\lambda) \theta = \sigma_{k-1}(\lambda).
\]

6. Remarks on Uhlhorn's theorem. The following remarks apply to the complex as well as the quaternion case. Uhlhorn has obtained the very interesting result that Wigner's theorem holds under considerably weaker assumptions. In terms of the conditions listed in Sec. 1.2 it suffices to maintain (a) and (d) while (b) is replaced by the condition (b') \( T e_1, T e_2 = 0 \) if and only if \( e_1 \cdot e_2 = 0 \) (preservation only of the transition probability zero!) On the other hand it is necessary to assume dim \( \mathcal{C} \geq 3 \).

Since, however, the condition (d)—or possibly some weaker substitute—is actually needed for the proof of Uhlhorn's result it seems to the writer that the main theorem proved in the present note retains an independent mathematical interest.

In conclusion it may be mentioned that a minor modification of Wigner's construction also yields a simple proof of Uhlhorn's theorem.