that the spin which we have found is the correct relativistic quantity as we encountered it in $\int d^3r\,(x^i\theta^{0j}-x^j\theta^{0i})$. This is the proper Lorentz angular momentum and formally, as well as in the vacuum sector, satisfies, together with $\int d^3r \times x^i\theta^{00}$, the Lorentz algebra. Moreover Goldhaber has recently shown that our half-integer-spin dyons obey Fermi-Dirac statistics.[10]

One of us (R. J.) is happy to acknowledge conversations with P. W. Anderson, who insisted on the possibility of contructing noninteger angular momentum in gauge theories. Both of us benifitted from discussions with S. Coleman and J. Goldstone.

---

*Work supported in part through funds provided by the U. S. Energy Research and Development Administration under Contract No. AT (11-1)-3069.

[1]T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York, 1969); A. C. T. Wu and T. T. Wu, J. Math. Phys. (N.Y.) 15, 53 (1974); G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)]; B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975); W. J. Marciano and H. Pagels, Phys. Rev. D 12, 1093 (1975); A. Charkabarti, Ann. Inst. Henri Poincaré 23, 235 (1975), and Nucl. Phys. B101, 159 (1975); E. Corrigan, D. I. Olive, D. B. Fairlie, and J. Nuyts, to be published.

[2]For a recent comprehensive review see S. Coleman, to be published.

[3]R. Jackiw and C. Rebbi, Phys. Rev. D (to be published).

[4]'t Hooft, Ref. 1; Polyakov, Ref. 1.

[5]Precise limits on the magnitudes are given by R. Rajaraman and E. Weinberg, Phys. Rev. D 11, 2950 (1975).

[6]We envision quantization procedures for the monopole advocated by J. Goldstone and R. Jackiw (unpublished); E. Tomboulis and G. Woo, to be published.

[7]J.-L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); E. Tomboulis, Phys. Rev. D 11, 1678 (1975); N. Christ and T. D. Lee, Phys. Rev. D 11, 1606 (1975); C. Callan and D. J. Gross, Nucl. Phys. B93, 29 (1975).

[8]We are repeating a classic calculation of the angular momentum in a magnetic monopole field; H. A. Wilson, Phys. Rev. 75, 309 (1949).

[9]It has been long recognized that spin and magnetic monopoles may be related. An early reference is A. S. Goldhaber, Phys. Rev. 140, B1407 (1965). Also we have learned that P. Hasenfratz and G. 't Hooft have arrived at conclusions similar to ours {following Letter [Phys. Rev. Lett. 36, 1119 (1976)]}.

[10]A. Goldhaber, second following Letter [Phys. Rev. Lett. 36, 1122 (1976)].

Fermion-Boson Puzzle in a Gauge Theory*

P. Hasenfratz
Theoretical Physics Department, University of Utrecht, Utrecht, The Netherlands

and

G. 't Hooft†
Physics Laboratories, Harvard University, Cambridge, Massachusetts 02138

(Received 1 March 1976)

It is argued that magnetic monopoles in an SU(2) gauge theory may bind with an ordinary boson with isospin, to give bound states with spin. If the isospin of the free boson is integer or half-odd-integer, the total angular mometum of the bound state is integer or half-odd-integer, respectively. According to the spin-statistics theorem we can obtain fermions this way in a theory that started off with bosons only.

Recently it was shown by Coleman[1] that in a two-dimensional boson theory (the massive sine-Gordon theory) "solitons" occur that actually obey Fermi statistics. We present here a theory in four space-time dimensions where fermions can arise in a similar manner.

Let us consider the Lagrangian[2]

$$\mathcal{L}=-\tfrac{1}{4}G_{\mu\nu}{}^aG_{\mu\nu}{}^a-\tfrac{1}{2}(D_\mu Q^a)^2-\tfrac{1}{8}\lambda(Q^2-F^2)^2, \qquad (1)$$

describing neutral massless photons, a massive charged vector boson, and a neutral Higgs boson. Here, λ and F are parameters, and

$$G_{\mu\nu}{}^a=\partial_\mu A_\nu{}^a-\partial_\nu A_\mu{}^a+e\epsilon_{abc}A_\mu{}^bA_\nu{}^c,$$
$$D_\mu Q^a=\partial_\mu Q^a+e\epsilon_{abc}A_\mu{}^bQ^c. \qquad (2)$$

This model has a magnetic monopole soliton, described by a classical solution of the form[3]

$$(Q^a)^{\rm cl}=r_aQ(r),\quad (A_i{}^a)^{\rm cl}=\epsilon_{ial}r_lW(r),$$
$$(A_0{}^a)^{\rm cl}=0,\quad a=1,2,3 \qquad (3)$$

We have the following asymptotic behavior:

$$Q(r) \xrightarrow[r\to\infty]{} F/r, \qquad W(r) \xrightarrow[r\to\infty]{} -1/er^2. \tag{4}$$

The electric field is zero and the magnetic field is

$$\vec{H} = \vec{r}/er^3.$$

Thus we have a magnetic charge,

$$g = 4\pi/e,$$

satisfying Schwinger's quantization condition.

We now introduce a new scalar (complex) multiplet of fields, transforming as some irreducible representation of SU(2). Thus we add to the Lagrangian

$$\Delta\mathcal{L} = -(D_\mu U)^* D_\mu U - V_2(U^*U, Q^2), \tag{5}$$

with

$$D_\mu U = \partial_\mu U - ieT^a A_\mu{}^a U, \qquad [T^a, T^b] = i\epsilon_{abc} T^c, \tag{6}$$

and for instance

$$V_2 = m^2 U^*U - hU^*U(F^2 - Q^2) + g(U^*U)^2. \tag{7}$$

Thus we have added to the system a multiplet of elementary particles, whose electric charges are either $-se, \dots, -e, 0, e, \dots, se$, or $-se, \dots, -\frac{1}{2}e, +\frac{1}{2}e, \dots, se$, depending on whether $2s+1$ is odd or even. In the latter case our monopole actually is a Dirac monopole, because

$$gq = 2\pi, \tag{8}$$

if $q = \frac{1}{2}e$ is the charge quantum. It is this case that we have in mind. It has been known for a long time that the Hamiltonian eigenstates of a charged boson in the presence of a magnetic point source have half-odd-integer angular momentum.[4] It is the purpose of our present Letter to show that this feature persists if the magnetic point source is replaced by the nonsingular soliton in the gauge theory and if the bound state is described as being one of the quantum excitations of the magnetic monopole. Assuming the spin-statistics theorem to be valid here we conclude that this quantum excitation is a fermion, in spite of the fact that the model has only bosonic elementary fields.

While these notes were written we became a aware of the work of Jackiw and Rebbi[5] who reached the same conclusions in the same model, but used a different approach. They considered the case that $U(x) \neq 0$ in the classical solution with lowest energy, after which they introduced a collective variable corresponding to isospin rotations of $U(x)$. We will take the case that the classical value of $U(x)$ is zero in the presence of the monopole solution, which is certainly compatible with the wave equations and also will minimize the energy if m and g in Eq. (7) are large enough. The functions $Q(r)$ and $W(r)$ in Eq. (3) are unaltered now. We quantize the theory in the one-soliton sector of Hilber space, writing

$$Q^a = (Q^a)^{\mathrm{cl}} + (Q^a)^{\mathrm{qu}}, \qquad A_\mu{}^a = (A_\mu{}^a)^{\mathrm{cl}} + (A_\mu{}^a)^{\mathrm{qu}}, \qquad U = U^{\mathrm{qu}}. \tag{9}$$

Collective variables are indispensible for describing momentum and charge conservation. (Pure isospin rotations can be dealt with by fixing the gauge.) The quantum excitations are then taken to be only those that are orthogonal to infinitesimal translations and charge gauge rotations. The infinitesimal fluctuations of Q^{qu} and A^{qu} do not mix with those of U^{qu}, and we will only consider the latter. The equations for the stationary modes ($\partial_0 \to -iE$), normalized to 1, can be considered as Schrödinger equations for a scalar particle with isospin T and mass m, interacting with the monopole. In the nonrelativistic limit

$$H^{\mathrm{op}} U^{\mathrm{qu}} = E U^{\mathrm{qu}}, \tag{10}$$

where

$$H^{\mathrm{op}} = (1/2m)[p_j{}^{\mathrm{op}} - e(A_j{}^a)^{\mathrm{cl}}(x)T^a]^2 + V(x), \tag{11}$$

with

$$p_j = (1/i)\partial/\partial x_j,$$

and $V(x)$, following from Eq. (7), may provide the binding force.

The symmetry properties of this equation follow from the form of the classical values of A and Q [Eq. (3)]. Rotational invariance implies that

$$\vec{J}^{\mathrm{op}} = (\vec{x} \times \vec{p}) + \vec{T} \tag{12}$$

is conserved. Its eigenvalues are clearly integer or half-integer if those of $\vec{T}$ are integer or half-integer, respectively. Another conserved quantity is electric charge, which is well defined far away from the center:

$$C \xrightarrow[|x|\to\infty]{} e(\vec{x}\cdot\vec{T})/|x|. \tag{13}$$

States are completely classified by C and J.

The fact that $\vec{J}^{\mathrm{op}}$ really describes total angular momentum of the bound state can be concluded in different ways:

(a) $\vec{J}^{\rm op}$ commutes with the Hamiltonian. The energy of the states depends only on J^2 and C. Thus there is a $2j+1$ degeneracy, following from the fact that the total system is rotationally invariant.

(b) $\vec{J}^{\rm op}$ satisfies the correct commutation rules:

$$[J_i{}^{\rm op}, J_j{}^{\rm op}] = i\epsilon_{ijk} J_k{}^{\rm op}. \tag{14}$$

It can be shown that

$$[J_i{}^{\rm op}, P_j] = i\epsilon_{ijk} P_k, \tag{15}$$

where P generates translations of the total system.

(c) It is enlightening to consider the classical limit of Eq. (11) in which

$$[p^{\rm op}, x] \to 0, \quad [T^a, T^b] \to 0, \tag{16}$$

etc., corresponding to large momenta, coordinates, masses, and large values for T^2. Replacing commutators by Poisson brackets, we get the following classical equations for the isospin particle in the neighborhood of the monopole:

$$\begin{aligned} \dot{x}_i &= (1/m)[p_i - e(A_i{}^a)^{\rm cl}(x)T^a], \\ \dot{p}_i &= (1/m)[p_j - e(A_j{}^a)^{\rm cl}(x)T^a](\partial/\partial x_i)(A_j{}^b)^{\rm cl} e T^b - \partial V/\partial x_i, \\ \dot{T}^a &= -\epsilon_{abc}(1/m)(p_i - eA_i{}^d T^d) e A_i{}^b T^c, \end{aligned} \tag{17}$$

which imply the Newton equation

$$m\ddot{x}_i = \dot{x}_k (G_{ik}{}^a)^{\rm cl} e T^a - \partial V/\partial x_i. \tag{18}$$

It is easy to show that the kinetic energy and the following quantities are conserved:

$$J_t = (\vec{x} \times m\dot{\vec{x}})_t + [T^t + WeT^b(x^2\delta_{tb} - x_t x_b)]. \tag{19}$$

The first term is the orbital angular momentum, while the second one—as will be shown—is the angular momentum which is stored in the Yang-Mills-Higgs field system in the presence of an isospin particle at $\vec{x}$ with isospin-charge eT^a. So $\vec{J}$ is the conserved total angular momentum. If we express $\vec{J}$ through the Hamilton variables we get back Eq. (12).

The calculations of the angular momentum stored in the Yang-Mills-Higgs fields consist of the following considerations (we do not here give all mathematical details). By putting a particle with isospin-charge vector eT^a at the point $\vec{r}$, a field $A_0{}^a$, and so a momentum density, is created, which has the form

$$\mathcal{T}_{0i} = G_{0k}{}^a G_{ik}{}^a + (D_0{}^{ab} Q^b)(D_i{}^{ac} Q^c) \tag{20}$$

(similarly as in electrodynamics where the Coulombic field $\vec{E}$ of a charge creates a momentum density $\sim \vec{E} \times \vec{B}$ around the monopole). The corresponding angular momentum is given by

$$L_t{}^{\rm field} = \int d^3x\, \epsilon_{tsi} x_s \mathcal{T}_{0i}. \tag{21}$$

Using the form of the soliton solution,[3] the integrand can be cast in the form

$$\begin{aligned} &-A_0{}^a(2eW^2 + e^2x^2W^3 + eQ^2 + e^2x^2WQ^2)(\delta_{at}x^2 - x_a x_t) + \partial_k\partial_k A_0{}^a(x^2\delta_{at} - x_a x_t)W \\ &- \partial_k A_0{}^a[W(x_t\delta_{ak} - x_a\delta_{tk}) - eW^2 x_a(\delta_{tk}x^2 - x_t x_k)] - \partial_k[\partial_k A_0{}^a(x^2\delta_{at} - x_a x_t)W]. \end{aligned} \tag{22}$$

The next step is to use the equation of motion for $A_0{}^a$:

$$D_i{}^{ab} G_{0i}{}^b - e\epsilon_{abc}(D_0{}^{bd} Q^d) Q^c = eT^a \delta^3(\vec{x} - \vec{r}). \tag{23}$$

The source term at the right-hand side is due to the isospin particle. It is then a straightforward calculation to get

$$L_t{}^{\rm field} = \int d^3x \{[T^t + eWT^a(x^2\delta_{at} - x_a x_t)]\delta^3(\vec{x} - \vec{r}) + (\text{total derivatives})\}. \qquad (24$$

The surface terms can be shown to be zero, and so we get

$$L_t{}^{\rm field} = T_t + eW(r)T^a(r^2\delta_{at} - r_a r_t) \tag{25}$$

which proves the statement.

So far, we only considered the "little group" (i.e., the subgroup of Lorentz transformations that leave the energy-momentum of the bound state invariant). We found invariance if the elements of the little group are associated with a gauge transformation [by mapping the two SO(3) groups into each other]. This leads to the conservation of the angular momentum (12).

What about the more general elements of the Lorentz group? They also affect the collective coordinates and thus also the fields Q^{cl} and A^{cl}. The gauge for the new Q^{cl} and A^{cl}, after a Lorentz boost, is essentially free, so that the formulation of the more general Lorentz transformations will be much more ambiguous, contrary to those of the little group for which we could keep Q^{cl} and A^{cl} fixed. One consequence of this complication is that although it is easy to tell what the spin of the particle is, by consideration of the little group, it will be hard to derive a relativistic wave equation such as the Dirac or Klein-Gordon equation for the composite particles.

Of course the theory is expected to be fully Lorentz invariant and unitary, at least in the perturbation expansion because shifts such as in our Eq. (9) are known not to affect these properties essentially, even after renormalization.

After this work was completed, Goldhaber[6] showed how to extend the relation between spin and statistics for particles with both electric and magnetic charge: When two dyons are obtained as bound states of magnetic poles and electric charges, then the wave equation for the two compound objects may violate the spin-statistics theorem, but it contains more Dirac strings than necessary. These Dirac strings can be transformed away by means of an ordinary gauge transformation, and then a new minus sign restores the spin-statistics relation for the two dyons. These arguments are expected to apply also to the bound states we discussed here.

One of us (G.'t H.) wishes to thank K. Cahill, S. Coleman, J. L. Gervais, R. Jackiw, C. Rebbi, and A. Goldhaber for discussions, and one of us (P. H.) wishes to thank A. Frenkel and P. Hrasko for discussions.

*Work supported in part by the National Science Foundation under Grant No. MPS75-20427.

†On leave from the University of Utrecht.

[1]S. Coleman, Phys. Rev. D 11, 2088 (1975).

[2]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 28, 1494 (1972).

[3]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[4]P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 133, 60 (1931); I. E. Tamm, Z. Phys. 71, 141 (1931); M. Fierz, Helv. Phys. Acta 17, 27 (1944); see also A. Frenkel and P. Hraskó, Central Research Institute for Physics, Budapest, Report No. KFKI-75-82 (to be published).

[5]R. Jackiw and C. Rebbi, preceding Letter [Phys. Rev. Lett. 36, 1116 (1976)].

[6]A. Goldhaber, following Letter [Phys. Rev. Lett. 36, 1122 (1976)].

Connection of Spin and Statistics for Charge-Monopole Composites*

Alfred S. Goldhaber
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794
(Received 15 March 1976)

An object composed of a spinless electrically charged particle and a spinless magnetically charged particle may bear net half-integer spin, but the wave function of two such clusters must be symmetric under their interchange. Nevertheless, a careful study of the relative motion of the clusters shows that this symmetry condition implies the usual connection between spin and statistics.

If magnetic monopoles exist, then classical physics already tell us that a system of pole g and electric charge q has an angular momentum of magnitude gq/c, directed from charge to pole. In quantum mechanics this spin adds to orbital and intrinsic angular momenta, so that, for $gq/\hbar c = (2n+1)/2$, an otherwise integer-spin system will have net half-integral total angular momentum.[1] This holds equally well in the SU(2) gauge field formulation of charge-pole interactions (in fact, this spin may be used to derive the gauge field[2]), as has recently been emphasized by Jack-