The energy momentum tensor

This is also a little exercise of inserting c at the correct places. We put c equal 1 for convenience and re-insert it at the end.

Recall the Euler equations for an ideal fluid with density $\rho(x^i,t)$ and velocity $v^i(x^j,t)$:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v^i)}{\partial x^i}=0$$

$$\frac{\partial p^j}{\partial t}+\frac{\partial(p^j v^i)}{\partial x^i}=(\text{Force density})^j=-\frac{\partial p}{\partial x^j}$$

where $p^j\equiv\rho v^j$ is the j-component of the momentum density and p is the pressure. The first equation (the continuity equation) expresses the conservation of mass, the next equation expresses that the change of (a component of) momentum per volume according to Newton's second law is equal to the force component per volume, i.e. minus the gradient of the pressure.

Using the continuity equation in the momentum equation, this latter can be written as

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho}\,\vec{\nabla}p, \tag{1}$$

while the continuity equation can be written as a current conservation:

$$\partial_\mu j^\mu=0, \qquad j^\mu=(\rho,\rho\vec{v}) \tag{2}$$

For an ideal fluid we have:

$$T^{\mu\nu}=p\eta^{\mu\nu}+(p+\rho)U^\mu U^\nu, \quad \partial_\nu T^{\mu\nu}=0, \tag{3}$$

where $U^\mu=\gamma(v)(1,v^i)$ is the four-velocity ($U_\mu U^\mu=-1$).

(4) Write out the equations explicitly for $\mu=0$ and for $\mu=i$, and show (using both equations) that the one for $\mu=i$ can be written as

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1-v^2}{p+\rho}\left(\vec{\nabla}p+\vec{v}\frac{\partial p}{\partial t}\right).$$

(5) Show that this equation reduces to (1) in the non-relativistic limit and that the equation for $\mu=0$ likewise reduces to (2) in the non-relativistic limit.

Problem set 2

constant acceleration, part I

Consider the equation of motion in SR for a point particle:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}, \quad \vec{p} = m_0\gamma\vec{v}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

Consider the situation where $\vec{F}$ points along the x-axis and has the constant value $F = m_0 g$. Assume that the velocity is zero at time $t = 0$.

(1) Show that the motion of the particle is hyperbolic:

$$\left(x + \frac{c^2}{g}\right)^2 - (x^0)^2 = \left(\frac{c^2}{g}\right)^2, \tag{4}$$

where $x^0 = ct$ and $y = z = 0$.

(2) Let I denote the initial system where the particle is at rest at $t = 0$. Show that the proper time (times c) of a clock following the accelerated particle is given by:

$$\tau = \frac{c^2}{g} \sinh^{-1}\left(\frac{gx^0}{c^2}\right), \tag{5}$$

i.e.

$$x^0 = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c^2}\right). \tag{6}$$

(3) Show that the transformation from an inertial system I', where the accelerated particle is at rest at proper time τ at $x' = 0$ (and $y' = z' = 0$) with $x^{0\prime} = 0$ to I is given by:

$$x = \frac{c^2}{g}\left(\cosh\left(\frac{g\tau}{c^2}\right) - 1\right) + x' \cosh\left(\frac{g\tau}{c^2}\right) + x^{0\prime} \sinh\left(\frac{g\tau}{c^2}\right) \tag{7}$$

$$x^0 = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c^2}\right) + x' \sinh\left(\frac{g\tau}{c^2}\right) + x^{0\prime} \cosh\left(\frac{g\tau}{c^2}\right) \tag{8}$$

Derivation of the geodesic equation

Define the following quantity:

$$S[x(\lambda)] = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\, L(x(\lambda), \dot{x}(\lambda)), \qquad \dot{x}(\lambda) \equiv \frac{\mathrm{d}x}{\mathrm{d}\lambda}. \tag{30}$$

S is a functional of the set of paths $x(\lambda)$. Assume now that

$$L(x, \dot{x}) = \sqrt{g_{ij}(x)\, \dot{x}^i \dot{x}^j}. \tag{31}$$

(6) Prove that for the choice (31) the functional S is independent of the parametrization of the path $x(\lambda)$, i.e. given one parametrization, S is unchanged if we change parametrization $\tilde{\lambda} = f(\lambda)$, $\dot{f}(\lambda) > 0$, and consider the new path $\tilde{x}(\tilde{\lambda}) = x(f^{-1}(\tilde{\lambda}))$ (of course this path describes the same set of points, but the curve is "traveled" in a different "speed" as a function of $\tilde{\lambda}$).

(7) Show that if we choose the parameter $s = \tilde{\lambda}$ as the length of the curve:

$$s(\lambda) = \int_{\lambda_1}^{\lambda} \mathrm{d}\lambda' \sqrt{g_{ij}(x(\lambda'))\, \dot{x}^i(\lambda') \dot{x}^j(\lambda')} \tag{32}$$

then the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \tag{33}$$

simplify and can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{ik}(x)\frac{\mathrm{d}x^k}{\mathrm{d}s}\right) = \frac{1}{2}\frac{\partial g_{kl}(x)}{\partial x^i}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^l}{\mathrm{d}s}, \qquad g_{kl}(x)\frac{\mathrm{d}x^k}{\mathrm{d}s}\frac{\mathrm{d}x^l}{\mathrm{d}s} = 1. \tag{34}$$

Problem set 6

geodesics in Rindler space

Let (x^0, x) denote Minkowski coordinates in a system of inertia I, and let (w^0, w) denote the Rindler coordinates of a system of reference R with constant acceleration g relative to I:

$$\begin{aligned} x &= \frac{c^2}{g}\left(\cosh\left(\frac{gw^0}{c^2}\right) - 1\right) + w\cosh\left(\frac{gw^0}{c^2}\right) \\ x^0 &= \frac{c^2}{g}\sinh\left(\frac{gw^0}{c^2}\right) + w\sinh\left(\frac{gw^0}{c^2}\right) \end{aligned}$$

As shown we have

$$ds^2 = dx^2 - (dx^0)^2 = dw^2 - (1 + gw/c^2)^2(dw^0)^2. \tag{43}$$

(1) Write down the equation of motion for a particle in free fall as a second order differential equation in w^0.

(2) Show that the solution for a particle starting at $w = \tilde{w}$ at time $w^0 = 0$ with velocity zero is:

$$w(w^0) = \frac{c^2}{g}\left[\left(1 + \frac{g\tilde{w}}{c^2}\right)\frac{1}{\cosh\frac{gw^0}{c^2}} - 1\right]. \tag{44}$$

(3) Calculate the velocity

$$v \equiv c\frac{dw}{dw^0}.$$

Find the maximum velocity of the particle and compare with the velocity of light (calculate that too). What happens for $w^0 \to \infty$?

(4) Calculate the proper time of the particle as a function of w^0 and $\tilde{w}$.

Geodesics on the rotating disk

The metric of spatial geometry of the rotating disk is given by:

$$ds^2 = dr^2 + \frac{r^2}{1 - r^2\omega^2/c^2}\,d\theta^2. \tag{45}$$

(5) Write down the equations for a geodesic.

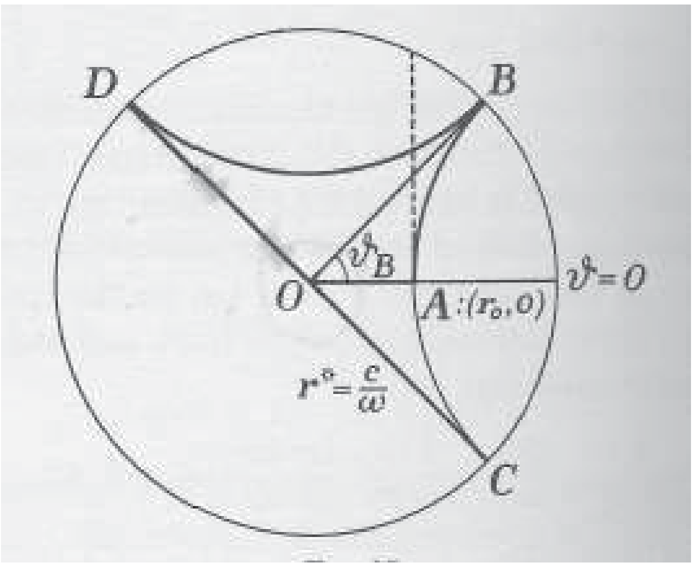

Figure 1:

(6) Show that first integrals are:

$$\frac{d\theta}{ds} = \alpha\,\frac{1 - r^2\omega^2/c^2}{r^2}, \tag{46}$$

$$\frac{dr}{ds} = \pm\sqrt{1 + \frac{\alpha^2\omega^2}{c^2} - \frac{\alpha^2}{r^2}}, \tag{47}$$

where α is an integration constant.

Thus

$$\frac{dr}{d\theta} = \pm\frac{r^2\sqrt{1 + \frac{\alpha^2\omega^2}{c^2} - \frac{\alpha^2}{r^2}}}{\alpha\left(1 - \frac{r^2\omega^2}{c^2}\right)} \tag{48}$$

and by integration we can in principle find $r(\theta)$.

(7) Consider the geodesic passing through $(r_0, 0)$ and having $dr/ds = 0$ at r_0. Find α expressed by r_0.

(8) Find the geodesics corresponding to the $\alpha = 0$.

(9) Find the angle ϕ between two geodesics which go through the same point expressed by α_1 and α_1 and the r-coordinate at the point where they meet. Answer:

$$\cos\phi = \pm\sqrt{1 + \frac{\alpha_1^2\omega^2}{c^2} - \frac{\alpha_1^2}{r^2}}\sqrt{1 + \frac{\alpha_2^2\omega^2}{c^2} - \frac{\alpha_2^2}{r^2}} + \frac{\alpha_1\alpha_2(1 - r^2\omega^2/c^2)}{r^2} \tag{49}$$

(10) Show that geodesics always met the boundary $r^* = c/\omega$ at a right angle (see figure).

(11) Show that the sum of angles of the triangle OAB in the figure is less than π. How small can the sum of the angles in a triangle formed by geodesics be ? How large ?

Problem set 8

The Riemann tensor

The Riemann tensor in n dimensions has at most $n^2(n^2-1)/12$ independent components.

(1) Show this using the symmetries of the Riemann tensor. (a) Show that $R_{\kappa\lambda\mu\nu}$ written as R_{AB}, $A=(\kappa\lambda)$ $B=(\mu\nu)$ is a symmetric matrix in the generalized indices A, B. A symmetric N-dimensional matrix has $N(N+1)/2$ independent elements. Show that the indices A and B can take $N=n(n-1)/2$ independent values. (b) From the number of independent elements calculated this way we have to subtract the constrains coming from the fact that the cyclic sum $R_{\kappa\lambda\mu\nu}+R_{\kappa\nu\lambda\mu}+R_{\kappa\mu\nu\kappa\lambda}$ is completely antisymmetric. Show that this gives $n(n-1)(n-2)(n-3)/4!$ constraints and obtain the total number $n^2(n^2-1)/12$ of independent components.

In two dimensions there is thus only one independent component and we can write:

$$R_{\kappa\lambda\mu\nu}=\frac{1}{2}R\left(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu}\right) \tag{53}$$

since this tensor has the right symmetries and the contraction gives the scalar curvature.

Assume now that the metric has the form

$$ds^2=dr^2+g_{\phi\phi}(r)d\phi^2. \tag{54}$$

We know already which components of Γ^i_{jk} are different from zero, expressed in terms of $g_{\phi\phi}(r)$ and $\partial g_{\phi\phi}/\partial r$.

(2) Show that $R^{\phi}{}_{r\phi r}=R/2$ and use the definition of $R^{\phi}{}_{r\phi r}$ in terms of Γ to find a general expression for R.

(3) Calculate $R/2$ for a sphere, for a pseudo-sphere and for the geometry corresponding to the rotating disk.

Consider the maximally symmetric spaces in n dimensions in the representation

$$Kx^ix^i+z^2=1, \qquad ds^2=dx^idx^i+\frac{1}{K}dz^2. \tag{55}$$

As we showed

$$g_{ij}(x)=\delta_{ij}+\frac{Kx^ix^j}{1-Kx^ix^i} \tag{56}$$

Problem set 9

The energy-momentum tensor

(1) Derive the energy-momentum tensor for a dust of point particles with action

$$S = \sum_n m_n \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}\frac{dx_n^\mu(\tau)}{d\tau}\frac{dx_n^\nu(\tau)}{d\tau}}$$

using the variation in $g_{\mu\nu}$.

(2) Derive the energy-momentum tensor for a scalar field with action

$$S = \int \mathrm{d}^4x \sqrt{-g(x)}\, \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right).$$

using the variation in $g_{\mu\nu}$.

(3) Derive the energy-momentum tensor for the electromagnetic field with action

$$S = \int \mathrm{d}^4x \sqrt{-g(x)}\, F_{\mu\nu}F^{\mu\nu}$$

using the variation in $g_{\mu\nu}$.

Use of the Lie-derivative

Under a diffeomorphism $x^\mu \to y^\mu(x^\nu)$ the metric will change as

$$g'_{\mu\nu}(y) = \frac{\partial x^\kappa}{\partial y^\mu}\frac{\partial x^\lambda}{\partial y^\nu} g_{\kappa\lambda}(x) \tag{60}$$

The change of the metric under an infinitesimal diffeomorphisms can be written as:

$$\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -g_{\mu\kappa}(x)\frac{\partial\varepsilon^\kappa(x)}{\partial x^\nu} - g_{\nu\kappa}\frac{\partial\varepsilon^\kappa(x)}{\partial x^\mu} - \frac{\partial g_{\mu\nu}(x)}{\partial x^\kappa}\varepsilon^\kappa(x), \tag{61}$$

for an infinitesimal diffeomorphism

$$y^\mu = x^\mu + \varepsilon^\mu(x). \tag{62}$$

Note that in (61) the tensor $g'_{\mu\nu}$ is evaluated in point x and *not* point y as in (60). This change of argument explains the presence of the last term in (61) and it is important in order to make $\delta g_{\mu\nu}(x)$ a tensor. $g'_{\mu\nu}(y) - g_{\mu\nu}(x)$ is not a tensor even for an infinitesimal diffeomorphism like (62)).