Physics Utrecht
EMMEΦ

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Quantum Black Holes, Firewalls

and the

Topology of Space-Time

A pedagogical treatment of a new approach

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Introduction

This is the 'Lecture Notes' version, and an extension, of my latest papers on black hole microstates. These notes highlight the essentials, such that they should be easier to read and understand. My plan is to keep this updated. If new insights arise, they will be added.

In version v3, a chapter, with figures, on "virtual black holes and space-time foam", was added at the end.
In version v4 further remarks were added, as well as several slides to explain better how some apparent contradictions are to be understood, describing in more detail the generic quantum state of a black hole, and how these evolve.
Version v5: More extended introduction, clarifications, responses to criticism, and finally some new features of the current algebra, in particular the relation between mode numbers such as ℓ and m, and the angular momentum.
Version v6: major revisions. Discreteness of the microstate spectrum explained. As I still noted inaccuracies in the earlier texts, I urge readers who hit upon any inconsistencies to mail the author: G.thooft@uu.nl.
Version v7: More texts on motivation, justification and procedure appeared to be needed.

These notes must be seen as a personal view of the author. Most importantly, we try to avoid any wild assumptions, as we usually see in the literature. Our claim is that everything here is essentially derived from basic laws that are well-established. Not even String Theory is used (this might change in the future).

These notes are not intended for use in talks (too much information on single "slides") with apologies for not always following the same notation as used elsewhere.

Black holes are an essential feature of the gravitational force. Without a thorough understanding of their properties one cannot understand quantum gravity. See: G. 't Hooft, arxiv:1612.08640 [gr-qc] + references there.

We emphasise that the approach explained here is not intended to *replace* other approaches, such as pure string theory descriptions or "fuzzballs". Rather, our approach is complementary. String descriptions, fuzzball descriptions and other attempts, all have in common that unconventional degrees of freedom are called upon because it is assumed that new 'Planck scale physics' should be necessary to address the information paradox. Alternatively, however, one could ask how *conventional physics* could be used to address this purported paradox. Why not consider the gravitational interactions between in- and out-going material? We find that this makes a big difference.

Hawking radiation consists predominantly of particles whose energies are far below the Planck value, and this is why one might attempt to formulate the effective laws of a black hole exclusively in terms of low-energy physics, of which the rules are well-documented: the Standard Model, augmented with perturbative quantum gravity. Thus, the Hilbert space of black hole states that we consider exclusively consists of a fixed background metric with excitations in the form of low-energy particles including gravitons. The gravitons represent large distance fluctuations of the background metric.

One might object that Planck scale phenomena should enter through the back door. To our astonishment however, we found this not to be the case. The physical degrees of freedom that are generated by the gravitational back reaction obey *linear equations that diagonalise*, when described in terms of the proper variables: the spherical harmonics.

If we would try to take into account matter that formed the black hole in the distant past, or matter in terms of which a black hole evaporates entirely, in the distant future, we would find that these degrees of freedom obtain energies far beyond the Planck scale, so we should *not* include these objects.

At first sight, our Hilbert space consisting only of low-energy particles ('soft particles') on a smooth background can only be maintained for a short amount of time as seen by the distant observer. But we made an amazing discovery: we can map states at (somewhat) later times 1-1 on states at earlier times, and by repeating this, we can reach all time epochs; a state in our Hilbert space *stays* in our Hilbert space.

At each *given* time, our procedure starts in a narrow time segment. Therefore, in that time segment, we should use the metric of a *permanent* black hole. The Penrose diagram for this *permanent* black hole has 4 regions, *I*, *II*, *III*, and *IV*.

There will be one novelty in our work: regions *II* and *IV* are identified as the *antipodes* of regions *I* and *III* (obtained by rotating 180° along the horizon).

The above *antipodal identification* is mandatory. Without it, our evolution law would not be unitary.

As soon as we adopt this constraint, as a new boundary condition for the eternal Penrose diagram, we find a set of eigen states of our evolution law. No further assumptions are now needed concerning the Planck scale.

The Planck scale does enter when we go to the high ℓ spherical modes, but, as the equations are linear and decouple (in the approximations used – and justified), we may include all ℓ up to a freely adjustable cut-off. The best way to choose this cut-off is by allowing exactly the number of microstates as is needed to reproduce Hawking's value.

Apart from the above, just known principles, such as Quantum Mechanics and General Relativity were used.

> *Note, that QM and GR are also assumptions, but they are much more compelling. Dropping or loosening these assumptions is much more difficult than keeping them as long as we can. No theory is sacred, and some sense of taste for what to keep and what not, is a useful attire for a theoretical physicist.*

Topics such as microstates can be understood very well in our theory. (As for the exact number of orthonormal microstates, our theory does allow us to count them, see arXiv:1809.05367)

In this work, we focus on the Schwarzschild black hole, but generalisation to Reissner-Nordström, Kerr, or Kerr-Newman seems to be straightforward. The BPS limit towards extreme black holes will require a bit more care.

The picture of quantum black holes given in several publications suggests it to be a chaotic system. Our work suggests that such chaos must be due to the use of inappropriate or ill-defined variables.

Note, that the 'fuzzy' or 'stringy' shape of the horizon in the 'fuzzball' theory assumes that the massive string modes are excited at the horizon. However, the massive string modes are much more energetic than the modes we need for our procedure. Thus, the states in Hilbert space that we consider, as long as ℓ values are not too high, should not be affected by such modes.

In these notes it will be found that quantum gravity in the bulk region up to very near the event horizons can be handled by standard perturbation expansion.

In Planck units, all out-going particles will have energies comparable to the *inverse* of the black hole mass. And of course, we can keep all in-going particles in that domain as well. Then all perturbations on the black hole's space-time metric will be small. To accommodate for these perturbations we include gravitons among the low energy particles.

Thus we obtain an S matrix that will be shown to be unitary *in the Hilbert space spanned by particles with energies (far) below the Planck mass* (we may choose the limit of ℓ so low that considerably fewer micro-states are involved, but if we do that, an other problem would arise, see end of slide 9).

Only at the horizon itself, one has to take into account the *gravitational back reaction*. Surprisingly, this entirely non-perturbative feature can be handled, since it happens to be described by linear equations.

Indeed, we emphasise that one can use ordinary coordinates in the bulk region, while at the (past as well as future) event horizons the gravitational back reaction *saves the theory from breaking down*.

What used to be thought of as chaos is now brought to order. Our system is comparable to a large number of quantum particles in a multi-dimensional rectangular box, with non-rational ratios in the lengths of the edges and the momenta of the particles. If viewed as a gas, it looks chaotic. But it is not.

In such a box, the particles do not interact, so all of their motions, in all directions, are decoupled. Every single particle, in every of the principal directions in the box, is described by a simple, one dimensional (quantum) equation of motion, which is trivial to solve for an undergraduate student. The black hole is just like this. Particles in the bulk interact, but only weakly, so, in our first approximations, we may neglect their interactions

Due to the fact that, on the horizons, we can expand the distribution of in- and out-going matter in *spherical harmonics*, we obtain equations that show that, on the horizons, all these different spherical harmonic modes completely decouple. Their equations factorise, just as the particles in our toy example, or the quantum modes in the hydrogen atom.

The gravitational back reaction also factorises. The great advantage of using this feature is that now we can easily solve the mathematical equations and see what happens.

At first sight, it seems that what we do is plainly wrong, and it was criticised as such in my email exchanges with colleagues. But the equations are so simple that one can investigate all alternatives.

It is essential to have Cauchy surfaces, and examine how the data on these surfaces evolve with time. What is the time coordinate? How can we ensure that the Cauchy data evolve through unitary equations? The Cauchy data should not get lost between the crevices of the horizons, which can easily happen while you think you are doing things correctly. So, the spherical wave expansion takes the place of the particle experiments that physicists could consult while working hard to construct what is now known as the Standard Model.

An important step has not yet been discussed. We explain how to obtain a unitary S matrix, expressed explicitly in terms of the orthonormal basis of a specially chosen Hilbert space. But this raises a question: *How should we map this orthonormal basis, defined in terms of the momentum distribution of in- and out-going matter, onto the orthonormal basis generated by the Fock space of the Standard Model* (or whatever replaces it at the scale where we consider the black hole)?

This mapping must be 1-1, but it is non-linear.

This problem seems to be very difficult. However, in ancient publications, the author demonstrated the close resemblance of our theory to a (rotated) string theory: the black hole horizon is mathematically closely related to a string theory world sheet. How do strings interact with particles coming in and out, at relatively low energies? Surprisingly, this difficulty is very similar to our difficulty with the 1-1 mapping sought for above. *And string theorists solved it!* The solution is that the state of N external particles is generated by putting N *vertex insertions* on the world sheet. Replace these vertex insertions by N Dirac delta peaks of in-going (or out-going) momentum, and then integrate over the positions of these vertices over the horizon.

Thus, we should copy the string theory solution into our horizon theory. It should be straightforward, but it has not been done explicitly. This time, we insist that the mapping should be a unitary one.

But yes, we have an important problem: there are practically no experiments. We conclude that we must be extremely careful and use devices such as spherical wave expansions or whatever else we can put our hands on, to get a clearer picture and to simplify our subject.

To my opinion, such advice should be followed meticulously.

See here how far I got. This author however, is still working mostly alone, and yes, I can make mistakes. If you, reader, think I made mistake(s) please do not hesitate to contact g.thooft@uu.nl

PART 1, Theoretical considerations and derivations

Our prototype is the Schwarzschild black hole. No serious complications are expected when generalised to Kerr-Newman or such. The *extreme* (BCS) black hole would not serve our purposes, because it is a limiting case, and its horizon is fundamentally different from the more generic black holes.

Consider the Schwarzschild metric:

$$\mathrm{d}s^2 = \frac{1}{1-\frac{2GM}{r}}\mathrm{d}r^2 - \left(1-\tfrac{2GM}{r}\right)\mathrm{d}t^2 + r^2\mathrm{d}\Omega^2 \; ; \qquad \begin{cases} \Omega & \equiv \quad (\theta, \varphi) \; , \\ \mathrm{d}\Omega & \equiv \quad (\mathrm{d}\theta, \, \sin\theta \, \mathrm{d}\varphi) \; . \end{cases}$$

Go to Kruskal-Szekeres coordinates $x, \ y,$ defined by

$$\begin{aligned} x\,y &= \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \quad ; \\ y/x &= e^{t/2GM} \quad . \end{aligned}$$

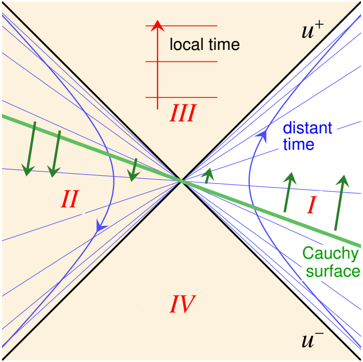
$$\mathrm{d}s^2 = \frac{32(GM)^3}{r}\, e^{-r/2GM}\, \mathrm{d}x\,\mathrm{d}y + r^2\mathrm{d}\Omega^2 \; .$$

At $r = 2GM$, we have $\quad x = 0$: past event horizon, and
$\quad y = 0$: future event horizon.

We concentrate on the region close to the horizon: $r \approx 2GM$. There:

$$x = \frac{\sqrt{e/2}}{2GM}\, u^+ \; ; \qquad y = \frac{\sqrt{e/2}}{2GM}\, u^- \; ; \qquad 2GM \equiv R$$

$$\mathrm{d}s^2 \;\to\; 2\mathrm{d}u^+\mathrm{d}u^- + R^2\mathrm{d}\Omega^2 \; . \qquad \text{time} \;\; t/4GM = \tau$$



$$u^-(\tau) = u^-(0)e^{\tau}$$
$$u^+(\tau) = u^+(0)e^{-\tau}$$

As time goes forwards, u^+ approaches the horizon asymptotically;

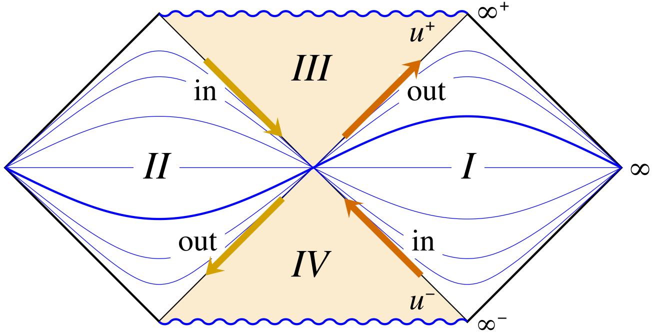
as time goes backwards, u^- approaches the past horizon asymptotically (tortoises).

Particles going in generate wave functions on u^+, particles going out start as wave functions on u^-.

If we redefine $u^+ \to f(u^+)$, $u^- \to g(u^-)$, then the metric keeps the form $\mathrm{d}s^2 = 2A(u)\mathrm{d}u^+\mathrm{d}u^- + r^2(u)\mathrm{d}\Omega^2$. $\rightarrow$ *Map $u^\pm$ on compact domains*:

The Penrose diagram.

For pure Schwarzschild (without matter either responsible for the formation of a black hole, or representing its final decay):



Now, we begin to deviate from standard practice:

This is the "eternal black hole" (it never formed, it will never decay, since, on the average, as many particles are going in, as there are going out). We keep it. Why?

We put the "microstates" on a Cauchy surface. As yet, we are only interested in how they evolve for a short amount of time. For the quantum properties of the BH, the distant past, and the distant future, should be irrelevant.

An elegant way to justify this: in case the implosion took place infinitely long ago, the out-going Hawking particles will be only in one state, the Hartle-Hawking state (see Slide # 51). When we observe the out-going Hawking particles, we project this state along a much larger class of basis states. Extrapolating these states back to the past, we will see many more recent in-going particles, including more recent implosions. We do allow all these states,

but we identify them by exclusively counting the recent in- and out-going particles only,

which will be further justified a posteriori.

This picture, however, is yet to be considered as an intermediate state of our discussion. We use it as a starting point to describe small variations in the black hole state during small periods in time. Later, we will address the black hole's long-term time dependence more accurately, and find the above formalism to be perfectly fitting for our procedure.

Hard and soft particles

First, consider the particles near the horizon(s), having momenta $(p^+,\, p^-,\, \tilde{p})$.
The mass shell condition is $2p^+p^- + \tilde{p}^2 + \mu^2 = 0$.
Here, $\tilde{p}$ is the transverse part of the momentum, $\mu =$ mass;
$|\tilde{p}| \approx L/R$ and μ are basically constant.
But $p^-(\tau) = p^-(0)e^{\tau}$, $p^+(\tau) = p^+(0)e^{-\tau}$ (See slide 13)
Define *soft* particles: $|\vec{p}|,\ \mu \ll M_{\text{Planck}}$ Negligible effect on space-time curvature.

Define hard particles as particles that do cause appreciable space-time curvature.
As $\tau \to \infty$, $p^+ \to 0$, $p^- \to \infty$: all in-particles will become *hard;*
As $\tau \to -\infty$, $p^- \to 0$, $p^+ \to \infty$: all out-particles originally were *hard.*

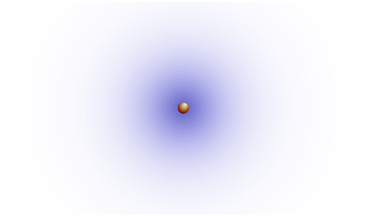
$|\tilde{p}|$ and μ stay small.

To understand what happens with the evolution at longer time intervals, we have to understand what the hard in- and out- particles do.

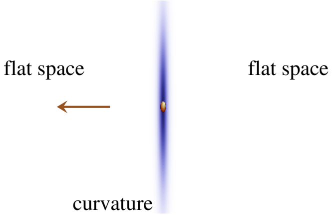
Consider an in-particle that is soft at $\tau \approx 1$. Now consider the same particle in a coordinate system centering at a later time $\tau \gg 1$. The in-particle then has become hard, since its momentum increases exponentially with time. Its interactions with other in-particles are negligible (they basically move in parallel orbits), but it does interact with the out-particles. The interaction through QFT forces stay weak, but the *gravitational forces* make that (early) in-particles interact strongly with (late) out-particles.

The *gravitational effect* of a fast, massless particle is easy to understand:

Schwarzschild metric of a particle with tiny rest mass $m \ll M_{\mathrm{Planck}}$:



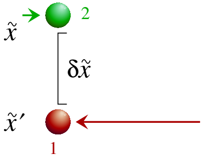
And now apply a strong *Lorentz boost*, so that $E/c^2 \gg M_{\mathrm{Planck}}$:

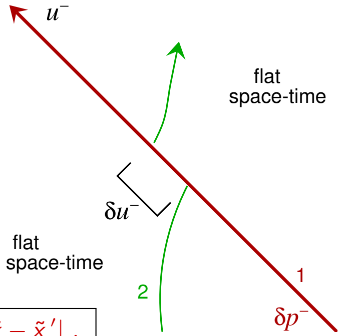


This we use to compute the gravitational force between hard and soft particles;

The gravitational back reaction:

Calculate the *Shapiro time delay* caused by the grav. field of a fast moving particle: simply Lorentz boost the field of a particle at rest:

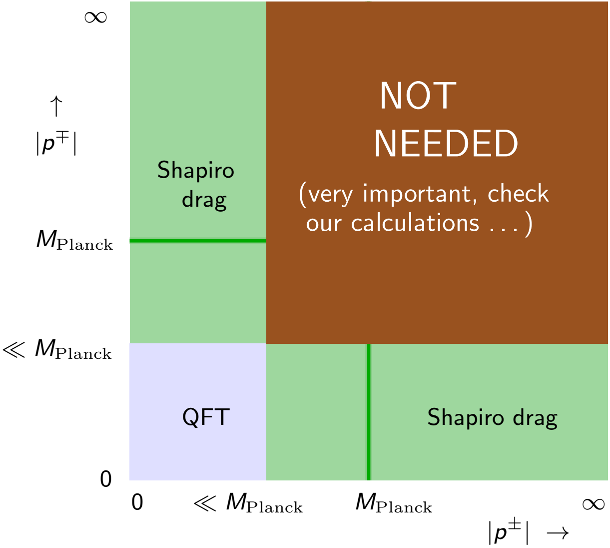




$$\boxed{\delta u^-(\tilde{x}) = -4G\, p^-(\tilde{x}\,')\ \log|\tilde{x} - \tilde{x}\,'|\ .}$$

P.C. Aichelburg and R.U. Sexl, J. Gen. Rel. Grav. **2** (1971) 303,
W.B. Bonnor, *Commun. Math. Phys.* **13** (1969) 163,
T. Dray and G. 't Hooft, *Nucl. Phys.* **B253** (1985) 173.

What we need to know is the interactions between in- and out-particles.



The fact that the mutual interactions between hard particles, at the Planck mass or beyond, will not be needed, is a very important aspect of this work. As you will see.

We start with *only soft particles* on a Cauchy surface of the Penrose diagram. These will define *all* quantum microstates of the black hole *at a given time*.

Now, the question is how do these evolve with time.
The soft particles won't stay soft; their longitudinal momenta will quickly explode. To see what happens, calculate δu^- (the Shapiro shift).

For that, we use Slide $\#$ 19, adapted to the Schwarzschild metric. This means that we replace the transverse coordinates $\tilde{x}$ by the solid angle $\Omega = (\theta, \varphi)$.

An in-particle with momentum p^- at solid angle $\Omega' = (\theta', \varphi')$ causes a shift δu^- at solid angle $\Omega = (\theta, \varphi)$:

$$\delta u^-(\Omega) = 8\pi G\, f(\Omega, \Omega') p^- \ ; \qquad (1 - \Delta_\Omega) f(\Omega, \Omega') = \delta^2(\Omega, \Omega') \ .$$

If there are many in-particles:

$$p^-(\Omega) = \sum_i p_i^- \delta^2(\Omega, \Omega_i)$$

$$\delta u^-(\Omega) = 8\pi G \int \mathrm{d}^2\Omega' f(\Omega, \Omega') p^-(\Omega') \ .$$

Our "reference" black hole will have a few soft particles seen by a *local* observer. This quantum state will be close to the Hartle Hawking state (see slide 51).

The *distant* observer will see unending streams of in- and out-particles with given positions u^+ or u^-. Suppose now that $p^-(\Omega)$ represents *all* in-particles needed to describe any black hole in a given quantum state.

This means that we replace δu^- by u^- itself.

Later, we will see how imploding matter may be described starting with momentum distributions p^- at the distant past.

Then the out-particles will be at positions $u^-(\Omega)$ given by

$$u^-_{\rm out}(\Omega) = 8\pi G \int {\rm d}^2\Omega' f(\Omega,\Omega') p^-_{\rm in}(\Omega') \quad .$$

Notice that *very early* in-going particles have huge values for p^-. Therefore they generate out-particles with huge values of u^- – these must have separated from the black hole long ago! We return to this subject later.

Spherical wave expansion:

$$\begin{aligned}
u^\pm(\Omega) &= \sum_{\ell,m} u_{\ell m} Y_{\ell m}(\Omega) \ , & p^\pm(\Omega) &= \sum_{\ell,m} p^\pm_{\ell m} Y_{\ell m}(\Omega) \ ; \\
[u^\pm(\Omega),\ p^\mp(\Omega')] &= i\delta^2(\Omega,\ \Omega') \ , & [u^\pm_{\ell m},\ p^\mp_{\ell' m'}] &= i\delta_{\ell\ell'}\delta_{mm'} \ ; \qquad (1) \\
u^-_{\rm out} &= \frac{8\pi G}{\ell^2+\ell+1} p^-_{\rm in} \ , & u^+_{\rm in} &= -\frac{8\pi G}{\ell^2+\ell+1} p^+_{\rm out} \ ,
\end{aligned}$$

$p^\pm_{\ell m}$ = total momentum in of $^{\rm out}_{\rm in}$-particles in (ℓ, m)-wave ,

$u^\pm_{\ell m}$ = (ℓ, m) - component of c.m. position of $^{\rm in}_{\rm out}$-particles .

Because we have linear equations, all different ℓ, m waves decouple, and for one (ℓ, m)-mode we have just the variables $u^{\pm}$ and $p^{\pm}$. They represent only one independent coordinate u^{+}, with $p^{-} = -i\partial/\partial u^{+}$.

The dynamics completely factorises in (ℓ, m) spherical harmonics.

This complete decoupling is due to our approximations, which are very good as long as $\ell \ll M_{\rm BH}$ ... in Planck units.

This remains true in the harmonics of a Kerr black hole, but that generalisation is not considered here.

3 more steps to be taken:

1. Starting with the QFT states of the particles entering the future event horizon, we must calculate their $p^-(\theta,\varphi)$ distribution there. The high momentum cut-offs must be chosen such that this mapping is unitary (or: reversible).
This is difficult.

Comparable to vertex insertions as in string theory.

2. $p^-(\theta,\varphi)$ on the future event horizon generates $u^-(\theta,\varphi)$ on the past event horizon. But its support is $[-\infty,\,+\infty]$, so it is spread over both regions *I* and *II*. What is the physical interpretation of region *II* ?

Postulate that region *II* refers to the same black hole as region *I*, but *not at the same solid angle* $\Omega=(\theta,\varphi)$. Only <u>one</u> possibility:

The antipodal identification: $\begin{array}{ccc} I & \leftrightarrow & II \\ \Omega & \leftrightarrow & \tilde{\Omega}=(\pi-\theta,\,\varphi+\pi) \end{array}$

3. The "firewalls". Soft particles become hard particles. Must be 'removed'.

Obvious suggestion: all information carried by the in- particles is now present in the out-particles. The in-particles are redundant ("quantum clones"). Leave the hard ones out. Hilbert space is completely specified by the coordinates $u^-(\Omega)$ of the out-particles, as soon as $|u^-|>L_{\mathrm{Planck}}$, these out-particles are then soft.

This is the "firewall transformation"; it removes all hard particles in the longitudinal direction, and thus removes the firewalls.

The antipodal transformation is the constraint that a space-time operation acts as an identity. This operation has $C = P = +1\,,\ T = CPT = -1$, as is explained in slide # 42.

The basic, explicit, calculation

The algebra (1), slide 23, generates the scattering matrix, by giving us the *boundary condition* that replaces $|\mathrm{in}\rangle$-states by $|\mathrm{out}\rangle$-states. This boundary condition replaces the old *brick wall model* and, in the spherical harmonics expansion, it is embarrassingly easy to derive

All of this is NOT a model, or a theory, or an assumption, but a <u>calculation</u> . . .

> Apart from the most basic assumption of unitary evolution,
> which forces us to fold the Penrose diagram along the antipodes,
> this is nothing more than applying GR and quantum mechanics !

At some points in the following derivations, it may seem that choices were made as to how to continue, such as the eternal BH Penrose diagram and the antipodal identification, and some of my choices are being criticised as they may seem illogical.
I emphasise however that they will all be vindicated by the final result, a unitary evolution law. There is no other way to obtain that. The internal logic will become clear. None of the other approaches in the literature that I have seen produce such explicit results.

At every (ℓ, φ), we have just one variable p^-, proportional to one u^-, with the Fourier transform p^+, proportional to u^+.

So we have 1 dimensional quantum mechanics !

Now let us deduce what happens with the Fourier transformation when we use tortoise coordinates.

Let there be two operators, u and p, obeying the commutator equation

$$[u,\, p] = i \ , \qquad \text{so that} \qquad \langle u|p\rangle = \tfrac{1}{\sqrt{2\pi}}\, e^{ipu} \ .$$

and a wave function $|\psi\rangle$, defined by $\psi(u) \equiv \langle u|\psi\rangle$. Its Fourier transform is

$$\hat\psi(p) \equiv \langle p|\psi\rangle = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}u\, e^{-ipu}\psi(u) \ .$$

Now introduce tortoise coordinates, and split both u and p in a positive part and a negative part:

$$u \equiv \sigma_u\, e^{\varrho_u} \ , \quad p = \sigma_p\, e^{\varrho_p} \ ; \quad \sigma_u = \pm 1 \ , \quad \sigma_p = \pm 1 \ , \quad \text{and}$$

$$\tilde\psi_{\sigma_u}(\varrho_u) \equiv e^{\frac12 \varrho_u}\, \psi(\sigma_u\, e^{\varrho_u}) \ , \quad \tilde{\hat\psi}_{\sigma_p}(\varrho_p) \equiv e^{\frac12 \varrho_p}\, \hat\psi(\sigma_p\, e^{\varrho_p}) \ ;$$

normalisation requires: (2)

$$|\psi|^2 = \sum_{\sigma_u=\pm} \int_{-\infty}^{\infty} \mathrm{d}\varrho_u |\tilde\psi_{\sigma_u}(\varrho_u)|^2 = \sum_{\sigma_p=\pm} \int_{-\infty}^{\infty} \mathrm{d}\varrho_p |\tilde{\hat\psi}_{\sigma_p}(\varrho_p)|^2 \ .$$

What is the Fourier transform in these tortoise coordinates?

$$\tilde{\tilde{\psi}}_{\sigma_p}(\varrho_p) = \sum_{\sigma_u=\pm 1} \int_{-\infty}^{\infty} \mathrm{d}\varrho_u \, K_{\sigma_u\sigma_p}(\varrho_u + \varrho_p)\, \tilde{\psi}_{\sigma_u}(\varrho_u) \ ,$$

$$\text{with} \quad K_\sigma(\varrho) \equiv \tfrac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\varrho}\, e^{-i\sigma\, e^{\varrho}} \ .$$

Notice the symmetry under $\varrho_u \to \varrho_u + \lambda$, $\varrho_p \to \varrho_p - \lambda$, which is simply the symmetry $u \to u\, e^{\lambda}$, $p \to p\, e^{-\lambda}$, a property of the Fourier transform, a consequence of *time translation invariance w.r.t. the external observer,* and an invariance of our algebra.

We now use this symmetry to write plane waves:

$$\tilde{\psi}_{\sigma_u}(\varrho_u) \equiv \breve{\psi}_{\sigma_u}(\kappa)\, e^{-i\kappa\varrho_u} \quad \text{and} \quad \tilde{\tilde{\psi}}_{\sigma_p}(\varrho_p) \equiv \breve{\breve{\psi}}_{\sigma_p}(\kappa)\, e^{i\kappa\varrho_p} \quad \text{with}$$

$$\breve{\breve{\psi}}_{\sigma_p}(\kappa) = \sum_{\sigma_p=\pm 1} F_{\sigma_u\sigma_p}(\kappa)\breve{\psi}_{\sigma_u}(\kappa) \ ; \quad F_\sigma(\kappa) \equiv \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_\sigma(\varrho) e^{-i\kappa\varrho} \mathrm{d}\varrho \ .$$

Thus, we see left-going waves produce right-going waves. One finds (just do the integral):

The Fourier transform in x, p space is non-local:

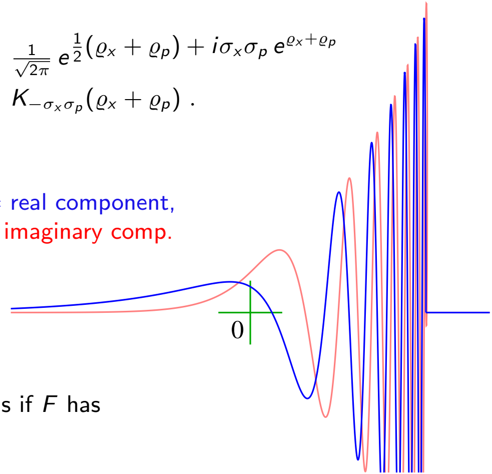
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}\, e^{ipx}$$

But if we write $x = \sigma_x\, e^{\varrho_x}$ and $p = \sigma_p\, e^{\varrho_p}$, where σ_x and σ_p are signs $\pm$, then the relation becomes:

$$\begin{aligned}\langle \varrho_x, \sigma_x | \varrho_p, \sigma_p\rangle &= \frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{2}(\varrho_x + \varrho_p) + i\sigma_x\sigma_p\, e^{\varrho_x+\varrho_p}} \\ &= K_{-\sigma_x\sigma_p}(\varrho_x + \varrho_p)\ .\end{aligned}$$

Blue = real component,
Red = imaginary comp.

$K_+(x)$:



In practice it will appear as if F has a finite support.

Look at how our soft particle wave functions evolve with time τ, slide # 13 or 16.

Their Hamiltonian is the dilaton operator. Let κ be the *energy:*

$$H = -\tfrac{1}{2}(u^+p^- + p^-u^+) \;=\; \tfrac{1}{2}(u^-p^+ + p^+u^-) =$$
$$i\frac{\partial}{\partial\varrho_{u^+}} \;=\; -i\frac{\partial}{\partial\varrho_{u^-}} \;=\; -i\frac{\partial}{\partial\varrho_{p^-}} \;=\; i\frac{\partial}{\partial\varrho_{p^+}} \;=\; \kappa\ .$$

The energy eigen states are $C\,(p^-)^{i\kappa} \;=\; C\,e^{i\kappa\,\varrho_{p^-}}$,

The Fourier operator on these states (σ is sign difference) is:

$$F_\sigma(\kappa) = \tfrac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\mathrm{d}y}{y}\,y^{\frac{1}{2}-i\kappa}\,e^{-i\sigma y} \;=\; \tfrac{1}{\sqrt{2\pi}}\,\Gamma(\tfrac{1}{2}-i\kappa)\,e^{-\frac{i\sigma\pi}{4}\,-\,\frac{\pi}{2}\kappa\sigma}\ .$$

Matrix $\begin{pmatrix} F_+ & F_- \\ F_- & F_+ \end{pmatrix}$ is unitary: $F_+F_-^* = -F_-F_+^*$ and $|F_+|^2 + |F_-|^2 = 1$.

The scattering matrix

Add the scale factor $\frac{8\pi G}{\ell^2+\ell+1}$, to get, if $u^\pm = \sigma_\pm e^{\varrho^\pm}$,

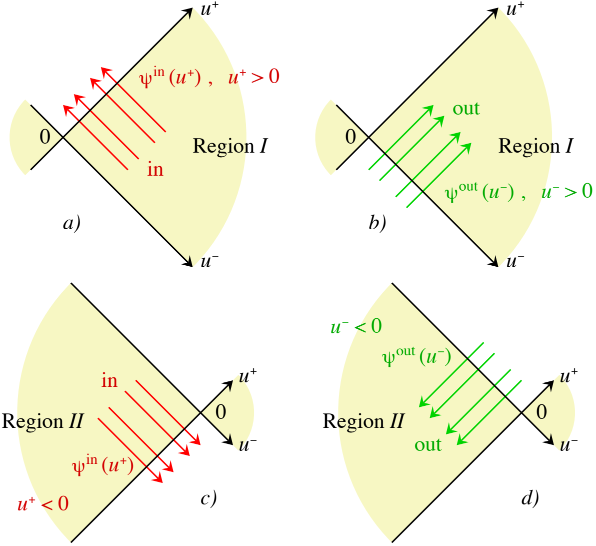
$$\psi^{\rm in}_{\sigma_+}\, e^{-i\kappa\varrho^+} \to \psi^{\rm out}_{\sigma_-} e^{i\kappa\varrho^-} \ , \tag{3}$$

$$\begin{pmatrix}\psi^{\rm out}_+ \\ \psi^{\rm out}_-\end{pmatrix} = \begin{pmatrix} F_+(\kappa) & F_-(\kappa) \\ F_-(\kappa) & F_+(\kappa)\end{pmatrix} e^{-i\kappa \log\left(8\pi G/(\ell^2+\ell+1)\right)} \begin{pmatrix}\psi^{\rm in}_+ \\ \psi^{\rm in}_-\end{pmatrix} \ .$$

These equations generate the contributions to the scattering matrix from all (ℓ, m) sectors of the system, where $|m| \le \ell$. At every (ℓ, m), we have a contribution to the position operators $u^\pm(\theta,\varphi)$ and momentum operators $p^\pm(\theta,\varphi)$ proportional to the partial wave function $Y_{\ell m}(\theta,\varphi)$. The signs of $u^\pm(\theta,\varphi)$ tell us whether we are in region I or region II. The signs of $p^\pm(\theta,\varphi)$ tell us whether we added or subtracted a particle from region I or region II.

In Slides # 43 – 44, we derive that, considering all $(\ell,\ m)$ values with $\ell < \ell_0 = \mathcal{O}(M_{\rm BH}/M_{\rm Planck})$ (the angular momentum limit), gives us $\approx e^{C^{\rm nst}\ \ell_0^2 \log \ell_0^2}$ microstates, which is the right order of magnitude. The microstates will turn out to be discrete, but note that this black hole is not exactly thermal, due to antipodal entanglement.

a) Wave functions $\psi(u^+)$ of the in-particles live in region I, therefore $u^+ > 0$.
b) Out-particles in region I have $\psi(u^-)$ with $u^- > 0$.



The physical picture

c, d) In region II, the in-particles have $u^+ < 0$ and the out-particles $u^- < 0$.

Note that the in-particles will never get the opportunity to become truly hard particles.

Eq. (4) is to be seen as a "soft wall"-boundary condition near the origin of the Penrose diagram. Wave functions going in are reflected as wave functions going out. These again emerge as soft particles.

Thus, there is no firewall, ever.

In the previous slide, the total of the in-particles in regions *I* and *II* are transformed (basically just a Fourier transform) into out-particles in the same two regions.

Note that the regions *III* and *IV* in the Penrose diagram (see slide 14) never play much of a role, even if an observer falling in region *III* would want to assure us that (s)he is still alive.

These regions are best to be seen as lying somewhere on the time-line *where time t is somewhere beyond infinity* (thus a mere repetition of the degrees of freedom we have seen before)

The catch?

Our "boundary condition at the origin" is in terms of momentum distributions $p^{\pm}(\theta, \varphi)$ and center-of-mass positions $u^{\pm}(\theta, \varphi)$ only.

But we would need the quantum wave functions as elements of Fock space (which would be specified by positions or momenta, but also by other quantum numbers!)

Such mapping should be unitary in Hilbert space. We are not certain that this can be done, but it is natural to look at how it is done in string theory. Our momentum distributions are like vertex insertions, although they are on the horizon instead of the string world sheet.

Presumably we are describing black holes with variable sizes. There may well be a cut-off at some large ℓ values depending on M. It can be argued that the microstates do form a *discrete set*, even though the wave functions seem to be continuous. This we explain In Slides $\#$ 43 – 44.

The antipodal identification

Regions *I* and *II* of the Penrose diagram are exact copies of one another. Often, it was thought that region *II* describes something like the 'inside' of a black hole. That cannot be right, since region *II*, like region *I*, has asymptotic regions. Hawking suggested that region *II* might be some other black hole, in an other universe, or far away in our universe. However, our 2×2 scattering matrix implies that the two regions are in contact with each other quantum mechanically. In ordinary branches of physics, such long-distance communication never takes place, and I don't think theories with such features make any sense.

It is far more natural to assume that region *II* describes the same black hole as region *I*. It must then represent some other part of the same black hole. Which other part? The local geometry stays the same, while the square of this $O(3)$ operator must be the identity.

There is exactly one possibility: This is the $O(3)$ operator $-\mathbb{I}$, which is:
the antipodal mapping.

It is a parity inversion, but later we shall combine it with a time inversion and a *C* inversion, so we have a *CPT* mapping, under which the laws of nature are invariant. A different, but complementary discussion is found on slide # 42.

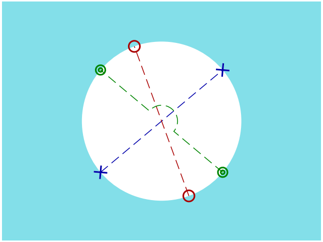
Antipodal *identification* only holds for the central point (origin) of the Penrose diagram. Regions *I* and *II* are different regions of the universe. But relating region *II* to region *I* by demanding that the angular coordinates are antipodes, means that now the mapping from Schwarzschild coordinates to Kruskal Szekeres coordinates is one-to-one. This now turns out to be an essential property of our coordinate transformations. Thus, we arrive at a new restriction for all general coordinate transformations:

> *In applying general coordinate transformations for quantised fields on a curved space-time background, to use them as a valid model for a physical quantum system, one must demand that the following constraint hold: the mapping must be one-to-one and differentiable. Every space-time point (r, t, θ, φ) now maps onto exactly one point $(x, y, \theta', \varphi')$, without the emergence of cusp singularities.*

The emergence of a non-trivial topology needs not be completely absurd, as long as no signals can be sent around much faster than the speed of light. This is the case at hand here.

Black emptiness: blue regions are the accessible part of space-time; dotted lines indicate identification.

The white sphere within is *not* part of space-time. Call it a 'vacuole'.

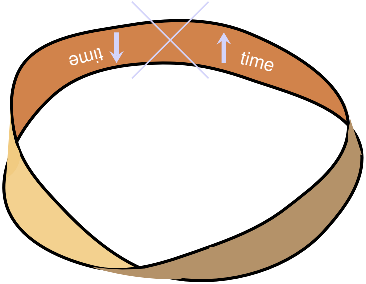


At given time t, the black hole is a 3-dimensional vacuole. The entire life cycle of a black hole is a vacuole in 4-d Minkowski space-time.

Space coordinates change sign at the identified points
– *and also time changes sign*
(Note: time stands still at the horizon itself).

Not that all $u^{\pm}$ and $p^{\pm}$ coordinates are odd when switching between antipodes. Therefore, only odd ℓ contribute in the spherical harmonic expansion.

A timelike Möbius strip



Draw a space-like closed curve:

Begin on the horizon at a point

$r_0 = 2GM \ , \ t_0 = 0 \ , \ (\theta_0, \varphi_0) \quad .$

Move to larger r values, then
travel to the antipode:

$r_0 = 2GM \ , \ t_0 = 0 \, , \ (\pi - \theta_0, \varphi_0 + \pi) \ .$
You arrived at the same point,
so the (space-like) curve is closed.

Now look at the environment $\{\mathrm{d}x\}$ of this curve. Continuously transport $\mathrm{d}x$ around the curve. The identification at the horizon demands

$$\mathrm{d}x \leftrightarrow -\mathrm{d}x \ , \quad \mathrm{d}t \leftrightarrow -\mathrm{d}t \ , .$$

So this is a Möbius strip, in particular in the time direction.
Note that it makes a T inversion and a CP inversion when going around the loop. CPT is preserved, herewith correcting an error in earlier versions of these notes.

There are no direct contradictions, but take in mind that

the local Hamiltonian density switches sign as well.

This is not true for the *total* Hamiltonian adopted by distant observers. locally, near the horizon, this is the *dilaton operator*. That operator leaves regions *I* and *II* invariant, and does not flip sign along the loop. Also, the boundary condition, our "scattering matrix", leaves this Hamiltonian invariant.

The S-matrix commutes with the Hamiltonian.

There are numerous treatises in the literature claiming solutions to the black hole information paradox, and about as many publications that dismiss these claims.
This author dismisses all claims, from both sides, that either ignore the gravitational back reaction of quantised excitations, or ignore the antipodal identification of points on the horizon – meaning that the horizon is a *projective* 2-sphere.

This concludes the most essential parts of the theory. If you have many burning questions, please contact g.thooft@uu.nl. Only if you are really interested in further details, problems and attacks aimed at solving them, consider PART 2, below.

PART 2, Further details and discussion

In the previous version of these notes, it was concluded that the antipodal identification requires CP symmetry conservation. This, we now argue, is not true.

The following argument on CP sounds contradictory to the one on slide # 36, but it really is equivalent:

First, the operation has $(x, y) \leftrightarrow (-x, -y)$, $(\mathrm{d}\theta, \mathrm{d}\varphi) \leftrightarrow (-\mathrm{d}\theta, \mathrm{d}\varphi)$, hence $PT = -1$, but PC can still be $+1$.

Secondly, when crossing a vacuole (see slide # 38), at a given, fixed time, say $\tau = 0$, one can say that a particle exchanges $r \leftrightarrow r$, but $\mathrm{d}r \leftrightarrow -\mathrm{d}r$: the particle exchanges front-to-back. Since furthermore $\theta \leftrightarrow \pi - \theta$, and $\varphi \leftrightarrow \varphi + \pi$, we have $\mathrm{d}\theta \leftrightarrow -\mathrm{d}\theta$, $\mathrm{d}\varphi \leftrightarrow \mathrm{d}\varphi$. Thus, of the local coordinates $\mathrm{d}r$, $\mathrm{d}\theta$, and $\mathrm{d}\varphi$, two change sign, so we *locally* have a $P = +1$ transformation. We can keep $C = +1$, so that $C = P = +1$ so neither P nor C symmetry are involved. Even while the antipodal identification has negative parity on the 2-dimensional horizon, where $r \equiv 2GM$, one can say that the antipodal identification, at fixed time, has positive 3-dimensional parity.

Thus, if now we add a $T = -1$ transformation in time, while $C = P = PC = +1$; $T = CPT = -1$. This is an absolute symmetry of any quantum field theory that can be used in the Standard Model.

Discreteness of the microstates

Now let us describe the full Hilbert space of a black hole. Let us split it up:

BH: The (classical) background metric, for simplicity the Schwarzschild one, *filled with only soft particles,* both in-going and out-going, but now we keep only those particles that, at a given, fixed, time τ are close to the horizon, typically inside a region $2M_{\rm BH} \le r \le 2M_{\rm BH}(1+\varepsilon)$, where ε is small but not very small, say $\varepsilon = 0.1$.

RU: The rest of the universe; all contributions of in- and out-particles which, at the time τ, have $r > 2M_{\rm BH}(1+\varepsilon)$.

The total mass-energy, of BH + RU, is conserved, but since the split is a function of time, the mass of the black hole is time dependent.

Let us take $\tau = 0$. According to Slide # 13, we then have $|x| = |y| \lesssim \sqrt{e\varepsilon}$, or $|u^\pm| \lesssim 2GM\sqrt{2\varepsilon}$. The fact that $u^\pm$ is taken to be inside this interval, implies that $p^\mp$ is discrete:

$$p^\mp = \frac{\pi\, n^\mp}{2GM\sqrt{2\varepsilon}} = u^\mp\,\frac{\ell^2+\ell+1}{8\pi G}\ .$$

Both u^+ and u^- are constrained, so that the integers $n^\pm$ are constrained:

$$|n^\pm| \le \varepsilon\,\frac{(\ell^2+\ell+1)GM^2}{\pi^2} \equiv n_0(\ell)\,\varepsilon\ .$$

$u^+(\ell, m)$ does not commute with $u^-(\ell, m)$, so we take either n^+ or n^- to characterise the state at (ℓ, m). The entire microstate can thus be characterised by the sequence

$$\left\{n^+(0,0), \cdots, n^+(\ell_{\max}, m_{\max})\right\}$$

(assuming some maximal values for (ℓ, m)).

The total number of states is (note that ℓ is odd):

$$N = \prod_{\ell,m}(2n_0(\ell)\,\varepsilon) = e^{\sum_{\ell=\mathrm{odd}}(2\ell+1)\log\left(n_0(\ell)\varepsilon\right)} = \mathcal{O}(e^{\ell_{\max}^2 \log \ell_{\max}})\,.$$

Note that our limits for the values of $u^\pm(\ell, m)$ for large ℓ can only indirectly be interpreted as particle positions since $u^\pm_{\ell,m}$ are partial wave amplitudes rather than coordinates.

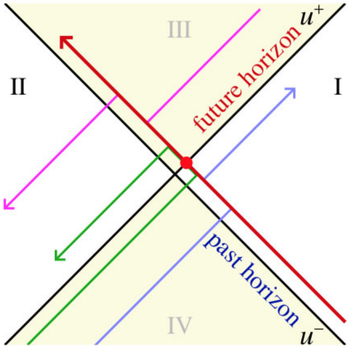
Assuming that $\ell_{\max} = \mathcal{O}(M_{\mathrm{BH}}/M_{\mathrm{Planck}})$, we recover the area law, but not the coefficient.

This topic must be further investigated.

Critical questions.

A bizarre sign switch

The effect of the gravitational footprint of an in-going particle (red line to upper left) onto the out-going particles (other coloured lines) is correctly accommodated for in this work. But there is an oddity, see our picture here of the allowed regions *I* and *II* and the unphysical regions *III* and *IV*.



The gravitational footprint moves the *pure states* of all out-particles combined in the light cone direction by an amount $\delta u^-(\theta, \varphi)$ in the same direction as the in-momentum $\delta p^-(\theta', \varphi')$.
This is a unitary transformation in the space of the combined wave functions only if all out-particles move by the same amount in the same direction, as is drawn here.

But this led to an apparently valid criticism:

This configuration does not appear to correspond to a valid solution of Einstein's equations for the local observer. What are we doing wrong?

It appears that this corresponds to a solution of Einstein's equations only if the momentum δp^- of the hard, in-particle flips its sign while passing through the future event horizon. At the same time, of course, the arrow of time switches sign.

Moreover, *those out-particles that are dragged right across the horizon (green line), do not seem to follow a geodesic at all, as regarded by the local observer.* Yet, of course, with our definition of the arrow of time, we have no choice, the particle has to continue its way as shown. What should a local observer say?

For the local observer, the diagram here does not show a classical solution, but it shows the action of an operator. The operator creates an extra in-particle (red line) that, in all states in regions *I* and *II*, shifts the geodesics of all out-particles by the same amount, in the same direction.

But if we represent the result of this operator as what happens later in time, then the fact that time runs backwards in region *II* leads to the impression that a discontinuity occurs on the other (future) horizon.

Operators should not be regarded as effects that agree with evolution equations; they represent what happens to a system if we make a change on the Cauchy surface, at some moment in time.

They subsequently affect the entire geodesics of the particles.

The change brought about here, has the effect of a positive δp^- particle in region *I* and at the same time the effect of a negative δp^- particle (an annihilated particle) in region *II*. Note that the operators $u^\pm$ and $p^\pm$ on the horizons all switch sign when passing from region *I* to *II*, since they are each other's antipodes, on the horizon.

The BMS group.

In a development parallel to our work here, it has been argued that "conservation of information" for black holes can be understood by using the BMS group. It was claimed that this group generates an infinite class of conserved charges that must be held responsible for safeguarding the information processed by the black hole. These charges are associated to supertranslations, and as such must take the form of (θ, φ)-dependent momenta in the light cone direction. To the present author, the physical interpretation of these arguments is less transparent. The arguments seem to be very formal, and thus suspect. I would like to see how the BMS group generates a unitary S-matrix.

It is quite conceivable that what is called BMS charges coincides with the momentum distributions that are central in our discussions. The functions we call $p^{\pm}(\theta, \varphi)$ do correspond to the entire momentum entering (p^{-}) pr leaving (p^{+}) at given solid angle $\Omega(\theta, \varphi)$ as accumulated over all time. So situating these "charges" on a projective square at $\mathcal{J}^{\pm}$ makes sense.

The black hole mass

Other question often asked: how does the mass of the black hole respond on the absorbed and emitted particles?

This is actually very easy. The energy (the sum of the numbers κ at every value of (ℓ, m)), is exactly conserved. To see how this works, arrange all in- and out- particles into three classes:

(*i*) The particles coming in when still far away from the future horizon, which are the firewall transforms of the out-particles that are still very close to the past event horizon, ready to emerge at some later era,

(*ii*) The soft particles in the parts of regions *I* and *II* that are in the neighbourhood, but not too close to any of the horizons, and

(*iii*) The particles moving out at a safe distance from the black hole, they are the firewall transforms of the particles that have arrived at a very close distance from the future event horizon.

The total energy, black hole plus these particles, is strictly conserved. But the particles of group (*i*) (the very late particles) and group (*iii*) (the very early particles) are so far away that we should not count those as part of the black hole mass/energy. The particles of group (*ii*) however, are so close to the horizon that these must be considered as part of the black hole.

We have to postulate that the entire black hole mass consists of the contributions of these particles.

Clearly, this mass is time dependent, as the distinction of a particle's classes is time dependent.

Note however, that a black hole is always surrounded by the cloud of entangled Hartle-Hawking particles.

The complete set of black hole states

When a local observer sees a vacuum, it means that the black hole is in the *Hartle-Hawking state,*

$$|\Omega\rangle = C \sum_{E,n} e^{-\frac{1}{2}\beta E} |E,n\rangle_I |E,n\rangle_{II} \ .$$

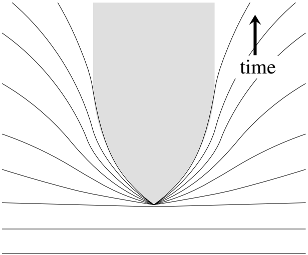
It is seen as a *vacuum* $|\Omega\rangle$ for the local observer, but contains states $|E,n\rangle$ in regions I and II for the global observer. C is a normalisation constant, and n represents all quantum numbers other than the energies E of the particle states – as seen by the global observer.

The *local* observer may now use his creation and annihilation operators to go to other states. All these states remain pure because of the antipodal identification, but we loose the entanglement of the above HH state.

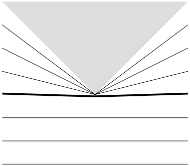
The moment of collapse and the moment of the final explosion

How does the antipodal identification get started when matter collapses into a black hole?
How does it end after the hole ended its life through evaporation?
This is a hard problem because at those moments the background metric is non stationary. Energy and momentum are ill defined.

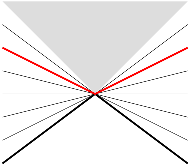
Penrose diagrams:



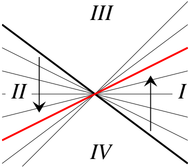
1) Realistic coordinates, with equal-time lines



2) situation at $t = 0$



3) situation at $t \gg 0$ showing Cauchy surfaces



4) antipodal identification

To obtain straight Cauchy surfaces, we need to invert time in region *II*, the antipode.

and now the grav. shift can be accounted for

Previous slide:

We describe here a metric with time-dependent source. the above was a suggestion for taking the back reaction into account, also during collapse. From picture (4) onwards, we propose to work just as in the “eternal” case.

When inspected in picture (1), we see that

> The antipodal identification also implies an inversion in the time direction.
> The entire inside region (grey) is removed,

so that only pure particle states can arise, also with gravitational back reaction. As in the eternal case, we replace the energetic in-particles at the past horizon by their footprints: the soft out-particles at the future event horizon.

The time-reverse of this, the final evaporation of the black hole, must be handled in exactly the same way.

More on the horizon algebra.

In my publications between $\sim$ 1985 and $\sim$ 1996, it was emphasised that the amplitudes produced by the mechanism of the gravitational back reaction, are strongly related to those of string theory. Perhaps it is string theory, although it seems more like a string theory with purely imaginary α'. The mathematical similarity originates simply from the fact that, as in string theory, we end up with algebraic relations of particle coordinates on a two-dimensional surface.

The string world sheet is replaced by the point of intersection of the future and the past event horizons. In- and out-going particles resemble the vertex insertions on a closed-string world sheet. Thus, one might expect an algebra akin to the Virasoro algebra. Yet I want to derive the black hole horizon algebra with utmost care, since the physical circumstances are quite different. If there is a relation, we need to find it, but also focus on differences.

In what follows we start producing some commutation rules, but calculations were only just begun.

For the time being, replace the spherical horizon(s) by an infinite, flat Rindler space-time. Replace the spherical harmonics $Y_{\ell\, m}(\Omega)$ by plane waves $e^{i\tilde{k}\cdot\tilde{x}}$, where the flat, transverse coordinates $\tilde{x}$ replace $\Omega = (\theta, \varphi)$ and $\tilde{k}$ replaces the "wave numbers" $(\ell,\, m)$.
We consider the case $\tilde{k}^2 \ll M^2_{\rm Planck}$ (This may later be relaxed).
Write the blue equations in slide 23 as the Fourier transforms

$$u^{\pm}(\tilde{x}) = \tfrac{1}{2\pi} \int \mathrm{d}^2\tilde{k}\, e^{\pm i\tilde{k}\cdot\tilde{x}}\, \hat{u}^{\pm}(\tilde{k})\ ,$$

$$p^{\pm}(\tilde{x}) = \tfrac{1}{2\pi} \int \mathrm{d}^2\tilde{k}\, e^{\pm i\tilde{k}\cdot\tilde{x}}\, \hat{p}^{\pm}(\tilde{k})\ ,$$

2-vectors are indicated by a tilde ($\tilde{}$). The inverse equations are

$$\hat{u}^{\pm}(\tilde{k}) = \tfrac{1}{2\pi} \int \mathrm{d}^2\tilde{x}\, e^{\mp i\tilde{k}\cdot\tilde{x}}\, \hat{u}^{\pm}(\tilde{x})\ ,$$

$$\hat{p}^{\pm}(\tilde{k}) = \tfrac{1}{2\pi} \int \mathrm{d}^2\tilde{x}\, e^{\mp i\tilde{k}\cdot\tilde{x}}\, \hat{p}^{\pm}(\tilde{x})\ ,$$

Signs $\pm$ are chosen for later convenience. We omit the carets ($\hat{}$) as their locations should be clear from the notation.

$u^{\pm}(\tilde{x})$ is gravitational displacement at $\tilde{x}$ caused by *all* $p_i^{\pm}$:

Important: the *momenta* $p^\pm$ are now handled as distributions, built up from deltas where particles enter;
the positions $u^\pm(\tilde{x})$ just mark the location of particles with respect to their horizons. Labelling the particles by an index i, we take:

$$p^\pm(\tilde{x}) \equiv \sum_i p_i^\pm \, \delta^2(\tilde{x}-\tilde{x}_i) \quad ; \qquad u^\pm(\tilde{x}_i) = u_i^\pm \ ,$$

$$u^-(\tilde{x}) = 8\pi G \sum_i f(\tilde{x}-\tilde{x}_i)\, p_i^- = \quad 8\pi G \int \mathrm{d}^2\tilde{x}'\, f(\tilde{x}-\tilde{x}')p^-(\tilde{x}') \ ;$$

$$\text{Green function obeys:} \qquad \tilde{\partial}^2 f(\tilde{z}) = \ -\delta^2(\tilde{z}) \ .$$

Our commutation rules are now:

$$[u_i^\pm,\, u_j^\pm] = [u_i^\pm,\, p_j^\pm] = [p_i^\pm,\, p_j^\pm] = 0 \ , \quad [u_i^\pm,\ p_j^\mp] = i\ \delta_{ij} \ ,$$

and the algebra turns into:

$$u^-(\tilde{k}) = \tfrac{8\pi G}{\tilde{k}^2}\, p^-(\tilde{k}) \quad ; \quad [u^\pm(\tilde{x}'),\, p^\mp(\tilde{x})] \ = i\,\delta^2(\tilde{x}'-\tilde{x}) \ .$$

$$[u^\pm(\tilde{k}),\, p^\mp(\tilde{k}')] = i\,\delta^2(\tilde{k}-\tilde{k}') \quad ; \quad [p^-(\tilde{k}),\, p^+(\tilde{k}')] \ = \tfrac{i\,\tilde{k}^2}{8\pi G}\,\delta^2(\tilde{k}-\tilde{k}') \ .$$

$$u^+(\tilde{k}) = -\tfrac{8\pi G}{\tilde{k}^2}\, p^+(\tilde{k}) \quad ; \qquad \tilde{\partial}^2\, u^\pm(\tilde{x}) \ = \ \pm\, 8\pi G\, p^\pm(\tilde{x}) \ .$$

Note Newton's G in denominator. This algebra is "non-perturbative" in gravity.

For future need:

$$[u^\pm(\tilde x),\, p^\mp(\tilde k)] = \tfrac{i}{2\pi}\, e^{\pm i\tilde k\cdot\tilde x}\ , \qquad [u^\pm(\tilde k),\, p^\mp(\tilde x)] = \tfrac{i}{2\pi}\, e^{\pm i\tilde k\cdot\tilde x}\ .$$

Because of the delta functions in the previous slide, the modes for the different harmonic waves $\tilde k$ all decouple. Formally, a generic quantum state (replacing Fock space for the Standard Model + perturbative gravity) is the product of states: $|u^+(\tilde k)\rangle$ for all allowed values of $\tilde k$.

Equivalently, one can label these states as $|u^-(\tilde k)\rangle$ or $|p^+(\tilde k)\rangle$ or $|p^-(\tilde k)\rangle$. These modes are like the one-particle states in QFT, but not quite, they label *matter waves*. Even so, we just have single wave functions $\psi(u^+)$ for a single coordinate u^+, at each value for $\tilde k$.

The wave functions $\psi(p^-)$, one for each value of $\tilde k$, are the Fourier transforms of $\psi(u^+)$, see slide 28. $\psi(p^\pm)$ is the same as $\psi(u^\pm)$, after scaling the variables $p^\pm$ to $u^\pm$, see previous slide.

It is important not to confuse the *wave modes* indicated by their *mode number* $\tilde{k}$ with the actual momentum this state might have. This will be indicated by the operator $\tilde{P} \equiv \int \mathrm{d}^2\tilde{k}\, \tilde{p}(\tilde{k})$, which equals $\sum_i \tilde{p}_i$ for all particles i. Thus, like $p^{\pm}$, $\tilde{p}$ is a distribution.

In order to calculate how each mode $\tilde{k}$ contributes an amount $\tilde{p}(\tilde{k})$ to the total transverse momentum, let us displace all harmonics by the same (infinitesimal) distance $\tilde{a}$. Then:

$$p^+(\tilde{k})\, e^{i\tilde{k}\cdot\tilde{x}} \rightarrow p^+(\tilde{k})\, e^{i\tilde{k}\cdot(\tilde{x}+\tilde{a})}\ , \quad \text{or:} \quad p^+ \rightarrow p^+ e^{i\tilde{k}\cdot\tilde{a}}\ ; \quad \delta p^+ = i\tilde{k}\cdot\tilde{a}\, p^+ \ \equiv i\tilde{a}\cdot\tilde{P}\ ,$$

and, noting that $u^{\pm}$ and $p^{\pm}$ are all in the same $\tilde{k}$ mode, we find that this mode's contribution to the momentum operator $\tilde{P}$ is:

$$\begin{aligned}\tilde{P} &= \tilde{k}\Big(p^+\frac{\partial}{\partial p^+} + \tfrac{1}{2}\Big) &&= -\,\tilde{k}\Big(p^-\frac{\partial}{\partial p^-} + \tfrac{1}{2}\Big) &&= -\,i\tilde{k}(\,p^+u^- + \tfrac{1}{2}i) \\ &= \tilde{k}\Big(u^+\frac{\partial}{\partial u^+} + \tfrac{1}{2}\Big) &&= -\,\tilde{k}\Big(u^-\frac{\partial}{\partial u^-} + \tfrac{1}{2}\Big) &&= -\,i\tilde{k}(u^-\,p^+ - \tfrac{1}{2}i)\end{aligned}$$

in agreement with the relations between $u^{\pm}$ and $p^{\pm}$ on slide 57.

What are the eigen values of $\tilde{P}$? To understand the question, we return to Hermitian operators (the plane waves are complex, hence non Hermitian).

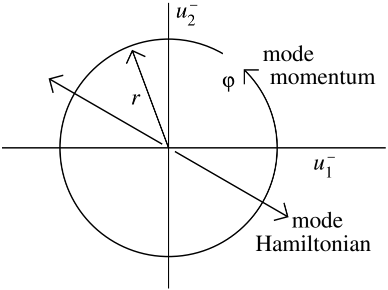
Split up $\quad e^{\pm i\tilde{k}\cdot\tilde{x}} = \cos(\tilde{k}\cdot\tilde{x}) \pm i\sin(\tilde{k}\cdot\tilde{x})$.
Write $\quad p^{\pm}(-\tilde{k}) = p^{\pm\,*}(\tilde{k})$, and

$$p^{\pm}(\tilde{k})\, e^{\pm i\tilde{k}\cdot\tilde{x}} + p^{\pm}(-\tilde{k})\, e^{\mp i\tilde{k}\cdot\tilde{x}} \equiv p_1^{\pm}\cos(\tilde{k}\cdot\tilde{x}) + p_2^{\pm}\sin(\tilde{k}\cdot\tilde{x}) \ ,$$
$$p^{\pm}(\tilde{k}) = p_1^{\pm} \pm i p_2^{\pm} \ ; \qquad u^{\pm}(\tilde{k}) = \tfrac{1}{2}(u_1^{\pm} \pm i u_2^{\pm}) \ .$$

The ½ is there to keep commutation rules as $[u_\alpha^{\pm},\, p_\beta^{\pm}] = i\delta_{\alpha\beta}$, where $\alpha,\,\beta = 1,2$. This gives the transverse momentum operator contribution of the two modes $\tilde{k}$ and $-\tilde{k}$ together:

$$\begin{aligned}\tilde{P} = -\tfrac{1}{2}\tilde{k}(u_1^- - iu_2^-)(p_1^+ + ip_2^+) + h.c. &= \tilde{k}(u_1^-\, p_2^+ - u_2^-\, p_1^+) \\ &= i\tilde{k}\frac{\partial}{\partial\varphi} \ .\end{aligned}$$

$$\tilde{P} = -\tfrac{1}{2}\tilde{k}(u_1^- - iu_2^-)(p_1^+ + ip_2^+) + h.c. = \tilde{k}(u_1^- p_2^+ - u_2^- p_1^+)$$
$$= i\tilde{k}\frac{\partial}{\partial\varphi}\ .$$



This generates a rotation in (u_1^-, u_2^-) space. The Hamiltonian, $H = u^- p^+ - \frac{1}{2}i = u_1^- p_1^+ + u_2^- p_2^+ - i$ generates a dilation (see slide 32)

The wave function for these two modes can be written as $\psi(u_1^-, u_2^-) = \psi(r, \varphi)$ (in polar coordinates), see Figure.

$\tilde{P}$ generates a *rotation* while H generates a *dilation*
(actually, a Lorentz transforation, if we take u_i^+ and u_i^- together)
Note that φ is periodic, end therefore the eigenvalues of $\tilde{P}$ are given by integers $n \in \mathbb{Z}$:

$$\boxed{\tilde{P} = \tilde{k}\, n} \,.$$

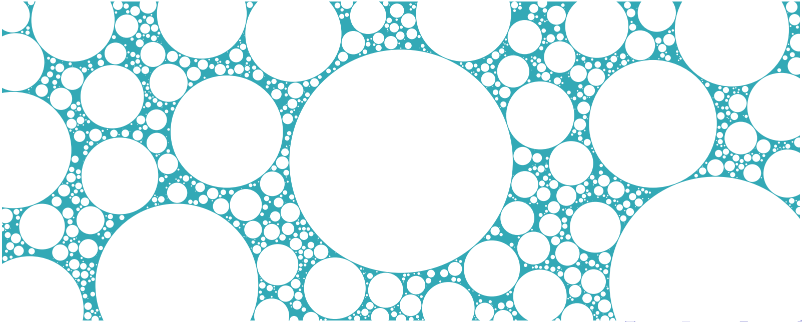
This illustrates that $\tilde{k}$ itself is *not* to be interpreted as transverse momentum;
transverse momentum is a multiple of $\tilde{k}$,

and similarly, (ℓ, m) are *not* to be interpreted as angular momenta.
Rather, they are spherical harmonic mode numbers.;
angular momentum is a multiple of ℓ.

We end with a speculation. It was introduced in version 3 of these slides.
Ignore it if you only want transparent physics.

Virtual black holes and space-time foam (Summary)

Virtual black holes must be everywhere in space and time. Due to vacuum fluctuations, amounts of matter that can contract to become black holes, must occur frequently. They also evaporate frequently, since they are very small. This produces small vacuoles in the space-time fabric.
How to describe multiple vacuoles is not evident. The emerging picture could be that of "space-time foam":



Work is still in progress. More than happy to discuss these ideas with like-minded colleagues.

The author benefited from discussions with P. Betzios, N. Gaddam, O. Papadoulaki, S. Mathur, L. Susskind, J. Maldacena and many others.

See: G. 't Hooft, arxiv:1612.08640 [gr-qc] + references there;
arxiv:1804.05744 [gr-qc];

See also:

P. Betzios, N. Gaddam and O. Papadoulaki, The Black Hole S-Matrix from Quantum Mechanics, JHEP 1611, 131 (2016), arxiv:1607.07885.

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N. Sanchez and B. Whiting, Quantum Field Theory and the Antipodal Identification of Black Holes, Nucl.Phys. **B283** (1987) 605.

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