SOLUTION OF THE STRING THEORY TEST

Tuesday, June 22, 2004

Exercise 1

1. *For an open superstring of type I, we have two sectors, called Ramond (R) and Neveu-Schwarz (NS). Summarize in your own words why we have exactly these two sectors.*
2. *Explain why the NS sector describes only bosonic solutions (string solutions that have integer spin) and the R sector only fermionic ones (solutions with spin integer+$\frac{1}{2}$).*
3. *In contrast with open strings, closed strings have four sectors: R-R, R-NS, NS-R and NS-NS. Explain why this is so, and explain which of these describe bosons and which describe fermions.*
4. *The GSO projection mechanism has two effects. Explain in your own words why this mechanism is needed to obtain consistent superstring theories.*
5. *Explain the main difference between type IIA and type IIB superstring theories.*

Solution

There is of course some amount of freedom in answering the theoretical questions of this exercise. However, for each point the following facts should at least be mentioned.

1. Varying the action for ψ^μ, one gets the following boundary conditions:

   $$(\psi_-^\mu \delta\psi_-^\mu - \psi_+^\mu \delta\psi_+^\mu)\Big|_{\sigma=0}^{\sigma=\pi} = 0\,,$$

   which, for an open string, can be solved by imposing $\psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0)$ (here we can choose the $+$ sign due to some freedom in the definition of the sign of the wavefunctions) and $\psi_+^\mu(\tau,\pi) = \pm\psi_-^\mu(\tau,\pi)$. Anti-periodicity is allowed since we are dealing with fermionic fields on the world-sheet, and more general phases are not allowed since the ψ^μ are real fields.

   The choice of the sign gives two different sectors of the theory, the Ramond sector ($+$ sign) and the Neveu-Schwarz sector ($-$ sign).

2. In the NS sector the frequencies of the oscillators are integers$+\frac{1}{2}$. There are no zero-modes, so the vacuum state is unique and must describe a bosonic spin

zero particle. All excited states are obtained by applying creation operators α^μ_{-n}, b^μ_{-r}, with $n, r > 0$. Since these operators are space-time vectors (carrying an index μ) they do not change the bosonic or fermionic nature of the states, so all states in the NS sector are bosonic.

In the R sector, the presence of the zero-mode oscillators d^μ_0, which commute with the string Hamiltonian and therefore do not change the energy of the state, gives the ground state a 16-fold degeneracy. This degeneracy is interpreted as space-time spin, since the zero-modes satisfy a Clifford algebra $\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}$. The R ground state is then fermionic, and all excited states are fermions as well since the creation operators α^μ_{-n}, d^μ_{-n} are all bosonic in space-time, as above.

3. In the closed string, boundary conditions for the fermions are chosen independently for left- and right-movers, and the spin of the resulting state (namely, the transformation properties under the Lorentz group) depends on the property of the product of the representations on the left and right. In summary we have:

NS-NS	boson	$\otimes$	boson	$=$	*boson*
NS-R	boson	$\otimes$	fermion	$=$	*fermion*
R-NS	fermion	$\otimes$	boson	$=$	*fermion*
R-R	fermion	$\otimes$	fermion	$=$	*boson*

4. The GSO mechanism projects out half of the states of the theory, so that in particular the tachyonic ground state in the NS-sector disappears, which is crucial for obtaining a consistent theory. Moreover, the number of bosonic and fermionic states of the theory after the projection is the same at each mass level, and the full theory can be shown to be space-time supersymmetric.

5. Type IIA and IIB superstring theories are defined by a different choice of the GSO projection in the right-moving Ramond sector. Keeping states with the same chirality on the left right gives type IIB theory, while choosing opposite chiralities yields type IIA. Both are consistent, supersymmetric, tachyon-free closed superstring theories with 2 gravitini (hence the name "type II"). The main difference is given by chirality: due to the choice of projection, type IIA is non-chiral while type IIB is chiral.

   The spectra of the two theories are of course different, as we can in particular see from the bosonic massless R-R states. Type IIA contains a vector C_1 and a three-form C_3, while type IIB contains C_0, C_2 and C_4.

Exercise 2

Consider the following parametric equations:

$$\begin{aligned} X^0 &= 3A\tau\,, \\ X^1 &= A\cos(3\tau)\cos(3\sigma)\,, \\ X^2 &= A\sin(\beta\tau)\cos(\gamma\sigma)\,, \end{aligned} \tag{2.1}$$

where A is a dimensionful constant and β and γ are arbitrary positive coefficients.

1. *Fix β and γ so that (2.1) describes an open string solution, fulfilling also the non-linear constraints $T_{\alpha\beta} = 0$ (in all the remaining parts of this exercise, always assume these values of β and γ). Write down the explicit expression of the solution in the form:*

$$X^\mu(\tau,\sigma) = X^\mu_L\,(\tau-\sigma) + X^\mu_R\,(\tau+\sigma)\,. \tag{2.2}$$

   Which boundary conditions does the solution fulfill in the various space-time directions?

2. *For what values of the modes x^μ, p^μ and α^μ_n does the general open string solution reproduce the expression (2.1)?*

3. *Plot the solution on the (X^1, X^2)-plane in time for τ varying from 0 to $2\pi/3$ with steps of $\pi/12$.*

4. *Compute the center-of-mass four-momentum P^μ and the angular momentum $J^{\mu\nu}$ for the solution under consideration, and show that they are conserved.*

5. *Consider now the compactification of the direction X^1 on a circle of radius R. We want to perform a T-duality transformation along X^1, which will yield a theory compactified on a circle of radius α'/R.*

   The new configuration is obtained by considering instead of X^1 the T-dual coordinate:

$$\widetilde{X}^1 = X^1_L\,(\tau-\sigma) - X^1_R\,(\tau+\sigma)\,. \tag{2.3}$$

   Does it still describe an open string solution fulfilling the energy-momentum constraints? Which boundary conditions along $\widetilde{X}^1$ does the T-dual solution obey? Can you give the T-dual solution an interpretation in terms of D-branes?

Solution

1. In order to be a string solution, (2.1) must satisfy the equations of motion $\partial^\alpha\partial_\alpha X^\mu = 0$. This is readily verified for X^0 and X^1, while for X^2 we find:

$$-\ddot{X}^2 + X^{2\prime\prime} = A\sin(\beta\tau)\cos(\gamma\sigma)(\beta^2-\gamma^2)\,,$$

so in order to make it vanish we need $\beta = \gamma$ (since both are positive). Another possible answer is saying that $\beta = \gamma$ is necessary for writing the solution in the form $X^2 = X^2_{\rm L}\,(\tau - \sigma) + X^2_{\rm R}\,(\tau + \sigma)$.

The $T_{\alpha\beta} = 0$ constraints can be written as:

$$(\dot{X}^\mu)^2 + (X^{\mu\prime})^2 = 0\,, \qquad \dot{X} \cdot X' = 0\,.$$

Starting from the first (with $\beta = \gamma$) we find:

$$\begin{aligned}
&(\dot{X}^\mu)^2 + (X^{\mu\prime})^2 \\
&\qquad = A^2\big[-9 + 9\sin^2(3\tau)\cos^2(3\sigma) + 9\cos^2(3\tau)\sin^2(3\sigma) \\
&\qquad\qquad + \beta^2\cos^2(\beta\tau)\cos^2(\beta\sigma) + \beta^2\sin^2(\beta\tau)\sin^2(\beta\sigma)\big]\,,
\end{aligned}$$

which vanishes if $\beta = 3$, in which case the second constraint can also easily seen to be satisfied. So we fix the coefficients to $\beta = \gamma = 3$.

We can write the solution isolating the left- and right-moving parts as follows:

$$\begin{aligned}
X^0 &= \frac{3A}{2}\left[(\tau - \sigma) + (\tau + \sigma)\right]\,, \\
X^1 &= \frac{A}{2}\left[\cos(3(\tau - \sigma)) + \cos(3(\tau + \sigma))\right]\,, \\
X^2 &= \frac{A}{2}\left[\sin(3(\tau - \sigma)) + \sin(3(\tau + \sigma))\right]\,,
\end{aligned} \tag{2.4}$$

We find $\partial_\sigma X^\mu\big|_{\sigma=0,\pi} = 0$, and therefore the solution satisfies Neumann boundary conditions at each endpoint (NN conditions) in every direction.

2. The general open string solution with NN boundary conditions can be written as:

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma)\,.$$

We therefore see that our solution is reproduced if we choose all mode coefficients to be zero except:

$$p^0 = \frac{3A}{2\alpha'}\,, \qquad \alpha^1_3 = -\alpha^1_{-3} = \frac{3A}{2i\sqrt{2\alpha'}}\,, \qquad \alpha^2_3 = \alpha^2_{-3} = \frac{3A}{2\sqrt{2\alpha'}}\,.$$

3. The plot describes a rigid open string which is folded three times over itself, and rotates of an angle of $\pi/4$ at each step of $\pi/12$ in time, coming back to the initial horizontal position at $\tau = 2\pi/3$.

4. The center-of-mass momentum is:

$$p^\mu = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\; \dot{X}^\mu\,,$$

so we obtain:

$$p^0 = \frac{3A}{2\alpha'}\,, \qquad p^1 = p^2 = 0\,.$$

The angular momentum is given by:

$$J^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\ \left[X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right]\,,$$

so we obtain:

$$J^{12} = -J^{21} = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\ 3A^2\cos^2(3\sigma) = \frac{3A^2}{4\alpha'}\,,$$

while all the other components vanish.

5. The T-dual coordinate is given by:

$$\begin{aligned}\widetilde{X}^1 &= X^1_{\mathrm{L}}\,(\tau-\sigma) - X^1_{\mathrm{R}}\,(\tau+\sigma)\\ &= \frac{A}{2}\left[\cos(3(\tau-\sigma)) - \cos(3(\tau+\sigma))\right] = A\sin(3\tau)\sin(3\sigma)\,,\end{aligned}$$

and one can easily see that it is still a consistent solution (T-duality is a duality among different theories mapping solutions to solutions).

$\widetilde{X}^1$ now satisfies $\partial_\tau \widetilde{X}^1\big|_{\sigma=0,\pi} = 0$, namely Dirichlet boundary conditions at both endpoints (DD) (mapping Neumann boundary conditions to Dirichlet is an important feature of T-duality for open strings). We then see that the open string endpoints are fixed in the direction $\widetilde{X}^1$, which means that (in the decompactification limit) the string is attached to a D$(D-2)$-brane, namely a D1-brane in our simple three-dimensional space-time.

Exercise 3

If we choose light-cone coordinates in a ten-dimensional target space-time spanned by X^μ ($\mu = 0, \dots, 9$) as follows:

$$X^{\pm} = \frac{X^0 \pm X^9}{2}\,, \tag{3.1}$$

we can write the standard Polyakov action in conformal gauge (namely, choosing the world-sheet metric $h_{\alpha\beta} = e^{\phi}\, \eta_{\alpha\beta}$) as:

$$S = -\frac{1}{4\pi\alpha'} \int d\tau\ d\sigma\ \left(-4\ \partial^\alpha X^+ \partial_\alpha X^- + \partial^\alpha X^i \partial_\alpha X_i\right)\,, \tag{3.2}$$

where $i = 1, \dots, 8$, and the world-sheet index α runs over the values $(0,1) \equiv (\tau, \sigma)$. We want now to study a slight modification of the above. Let us introduce a non-zero real parameter μ and consider the action:

$$S = -\frac{1}{4\pi\alpha'} \int d\tau\ d\sigma\ \left(-4\ \partial^\alpha X^+ \partial_\alpha X^- - \mu^2 X^i X_i\ \partial^\alpha X^+ \partial_\alpha X^+ + \partial^\alpha X^i \partial_\alpha X_i\right)\,, \tag{3.3}$$

which reduces to (3.2) in the limit $\mu \to 0$.

1. *Derive from the action (3.3) the equations of motion for the coordinate fields X^+, X^- and X^i.*

2. *Since the equation of motion for X^+ is unchanged from the flat space-time case, we are allowed to fix light-cone gauge:*

$$X^+ = \alpha' p^+ \tau\,. \tag{3.4}$$

   Rewrite the equations of motion for X^- and X^i in this gauge, and show that the fields X^i satisfy the equations of motion appropriate for 8 free massive scalars with a mass m given by:

$$m = \alpha' p^+ \mu\,. \tag{3.5}$$

3. *Derive the relation that the coefficients ω_n and k_n must satisfy for the following mode expansion of the "transverse" coordinate fields X^i to solve their equations of motion:*

$$\begin{aligned} X^i(\tau,\sigma) &= i\sqrt{2\alpha'}\ \frac{1}{m}\left[\alpha_0^i\ e^{-im\tau} - \tilde{\alpha}_0^i\ e^{im\tau}\right] \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{\omega_n}\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right]\,. \end{aligned} \tag{3.6}$$

   Verify that if:

$$k_n = n\,, \tag{3.7}$$

the solution (3.6) *satisfies the closed string boundary conditions:*

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma)\,. \tag{3.8}$$

What is the reason why the solution (3.6) *is conveniently expressed by means of zero-mode oscillators* α_0^i *and* $\tilde{\alpha}_0^i$ *instead of center-of-mass position and momentum?*

4. *Now consider the Hamiltonian:*

$$H = \int d\sigma \left(P_\mu \dot{X}^\mu - \mathcal{L} \right) \tag{3.9}$$

(a dot denotes the derivative with respect to τ*), where* $\mathcal{L}$ *is the Lagrangian density associated to the action, and* $P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$ *are the momenta conjugate to the fields* X^μ*.*

Show that, when choosing the light-cone gauge (3.4)*, the Hamiltonian* (3.9) *can be written in the following form:*

$$H = \int d\sigma \; \alpha' p^+ P_+ + H_{tr}\,, \tag{3.10}$$

where H_{tr} *depends only on the transverse fields* X^i*. Write down the explicit expression of* H_{tr}*.*

5. *By using the mode expansion* (3.6) *with* $k_n = n$*, write* H_{tr} *in terms of the oscillator modes* α_n^i *and* $\tilde{\alpha}_n^i$ *(neglecting any ordering problems). In the computation, choose the sign of* ω_n *to be the same as the sign of* n*.*

Solution

1. From the variation of the action we get the following equations of motion:

$$\begin{aligned}
\delta_{X^-} S = 0 \qquad &\rightarrow \qquad \partial^\alpha \partial_\alpha X^+ = 0\,,\\
\delta_{X^+} S = 0 \qquad &\rightarrow \qquad \partial^\alpha \partial_\alpha X^- + \frac{\mu^2}{2} \partial^\alpha \left[X^i X^i \partial_\alpha X^+ \right] = 0\,,\\
\delta_{X^i} S = 0 \qquad &\rightarrow \qquad \partial^\alpha \partial_\alpha X^i + \mu^2 X^i (\partial^\alpha X^+ \partial_\alpha X^+) = 0\,.
\end{aligned}$$

2. The first of the above equation is unchanged with respect to the flat-space case, and is trivially solved by choosing the light-cone gauge $X^+ = \alpha' p^+ \tau$. In this gauge the second and the third equations of motion (for X^- and X^i respectively) become:

$$\begin{aligned}
\partial^\alpha \partial_\alpha X^- - \mu^2 \alpha' p^+ X^i \partial_\tau X^i = 0\,,\\
\partial^\alpha \partial_\alpha X^i - m^2 X^i = 0\,,
\end{aligned}$$

where $m = \alpha' p^+ \mu$, as suggested in the text. The last equation is the Klein-Gordon equation describing 8 free massive scalars of mass m.

3. Let us compute some quantities which will be useful in this and subsequent points of the exercise:

$$
\begin{aligned}
X^i &= i\sqrt{2\alpha'}\ \frac{1}{m}\left[\alpha_0^i\ e^{-im\tau} - \tilde{\alpha}_0^i\ e^{im\tau}\right] \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{\omega_n}\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right] , \\
\dot{X}^i &= \sqrt{2\alpha'}\ \left[\alpha_0^i\ e^{-im\tau} + \tilde{\alpha}_0^i\ e^{im\tau}\right] \\
&\quad + \sqrt{2\alpha'}\sum_{n\neq 0}\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right] , \\
\ddot{X}^i &= -i\sqrt{2\alpha'}\ m\left[\alpha_0^i\ e^{-im\tau} - \tilde{\alpha}_0^i\ e^{im\tau}\right] \\
&\quad - i\sqrt{2\alpha'}\sum_{n\neq 0}2\omega_n\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right] , \\
X^{i\prime} &= -\sqrt{2\alpha'}\sum_{n\neq 0}\frac{k_n}{\omega_n}\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} - \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right] , \\
X^{i\prime\prime} &= -i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{2k_n^2}{\omega_n}\left[\alpha_n^i\ e^{-2i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^i\ e^{-2i(\omega_n\tau + k_n\sigma)}\right] .
\end{aligned}
$$

We then see that equation of motion $-\ddot{X}^i + X^{i\prime\prime} - m^2 X^i = 0$ is satisfied if the following relation holds:

$$
\omega_n^2 = k_n^2 + \frac{m^2}{4} .
$$

If $k_n = n$, all dependence on σ of X^i is of the form $e^{\pm 2in\sigma}$, which is of course invariant under $\sigma \to \sigma + \pi$. In this case, then, the solution satisfies the closed string boundary conditions $X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma)$.

The solution is conveniently expressed in terms of zero-mode oscillators (instead of position and momentum) since the fields X^i are massive, and are thus subject to a harmonic oscillator potential also for the center-of-mass modes.

4. The Hamiltonian in the light-cone coordinates we are using can be written as:

$$
H = \int d\sigma\left(P_+\dot{X}^+ + P_-\dot{X}^- + P_i\dot{X}^i - \mathcal{L}\right) .
$$

In the light-cone gauge $X^+ = \alpha' p^+ \mu$, we can immediately rewrite it as:

$$
H = \int d\sigma\ \alpha' p^+ P_+ + H_{\text{tr}} , \qquad \text{where} \qquad H_{\text{tr}} = \int d\sigma\left(P_-\dot{X}^- + P_i\dot{X}^i - \mathcal{L}\right) .
$$

We then compute:

$$
P_- = \frac{\partial\mathcal{L}}{\partial\dot{X}^-} = -\frac{\dot{X}^+}{\pi\alpha'} , \qquad P_i = \frac{\partial\mathcal{L}}{\partial\dot{X}^i} = \frac{\dot{X}^i}{2\pi\alpha'} .
$$

Substituting the above expression for the momenta and the lagrangian inside $H_{\rm tr}$, and imposing the light-cone gauge condition, one sees that the dependence on X^+ and X^- drops out and one is left with:

$$H_{\rm tr} = \frac{1}{4\pi\alpha'}\int d\sigma \left[(\dot{X}^i)^2 + (X^{i\prime})^2 + m^2(X^i)^2\right] .$$

5. *[This point of the exercise is not considered in the evaluation of the exam.]* One has to start with the expression of $H_{\rm tr}$ we just derived, and substitute into it all the relevant mode expansions written in the solution to point 3 of the exercise. The computation is somewhat lengthy, so we only summarize its main points.

   Start by writing $H_{\rm tr}$ as:

$$\begin{aligned} H_{\rm tr} = \frac{1}{2\pi}\int_0^\pi d\sigma \Bigg\{ & \Big[\alpha_0^i\, e^{-im\tau} + \tilde{\alpha}_0^i\, e^{im\tau} + \sum_{n\neq 0}\left(\alpha_n^i\, e^{-2i(\omega_n\tau - n\sigma)} + \tilde{\alpha}_n^i\, e^{-2i(\omega_n\tau+n\sigma)}\right)\Big] \\ \times & \Big[\alpha_0^i\, e^{-im\tau} + \tilde{\alpha}_0^i\, e^{im\tau} + \sum_{\ell\neq 0}\left(\alpha_\ell^i\, e^{-2i(\omega_\ell\tau-\ell\sigma)} + \tilde{\alpha}_\ell^i\, e^{-2i(\omega_\ell\tau+\ell\sigma)}\right)\Big] \\ + & \sum_{n,\ell\neq 0}\frac{n\ell}{\omega_n\omega_\ell}\left(\alpha_n^i\, e^{-2i(\omega_n\tau-n\sigma)} - \tilde{\alpha}_n^i\, e^{-2i(\omega_n\tau+n\sigma)}\right)\left(\alpha_\ell^i\, e^{-2i(\omega_\ell\tau-\ell\sigma)} - \tilde{\alpha}_\ell^i\, e^{-2i(\omega_\ell\tau+\ell\sigma)}\right) \\ - & m^2\Big[\frac{\alpha_0^i}{m}\, e^{-im\tau} - \frac{\tilde{\alpha}_0^i}{m}\, e^{im\tau} + \sum_{n\neq 0}\frac{1}{2\omega_n}\left(\alpha_n^i\, e^{-2i(\omega_n\tau-k_n\sigma)} + \tilde{\alpha}_n^i\, e^{-2i(\omega_n\tau+k_n\sigma)}\right)\Big] \\ \times & \Big[\frac{\alpha_0^i}{m}\, e^{-im\tau} - \frac{\tilde{\alpha}_0^i}{m}\, e^{im\tau} + \sum_{\ell\neq 0}\frac{1}{2\omega_\ell}\left(\alpha_\ell^i\, e^{-2i(\omega_\ell\tau-k_\ell\sigma)} + \tilde{\alpha}_\ell^i\, e^{-2i(\omega_\ell\tau+k_\ell\sigma)}\right)\Big]\Bigg\} \end{aligned}$$

   All terms proportional to $e^{2in\sigma}$ will vanish when integrated over σ. The only contributions will then come from terms which have no σ-dependence at all, which we summarize below.

   - *Zero-modes.* The zero-mode part has no dependence on σ, and one readily sees that the final result is a term $2\tilde{\alpha}_0^i\alpha_0^i$.
   - $n = \ell$. This contribution is proportional to:

     $$\alpha_n^i\tilde{\alpha}_n^i\left(1 - \frac{n^2}{\omega_n^2} - \frac{m^2}{4\omega_n^2}\right) ,$$

     which vanishes because of the relation $\omega_n^2 = n^2 + \frac{m^2}{4}$ we have previously derived.
   - $n = -\ell$. This is proportional to:

     $$\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right)\left(1 + \frac{n^2}{\omega_n^2} + \frac{m^2}{4\omega_n^2}\right) = 2\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n\right) .$$

Notice that for getting the above results it is crucial to define the sign of ω_n as suggested in the text, namely:

$$\omega_n = \operatorname{sgn}(n) \sqrt{n^2 + \frac{m^2}{4}} \, .$$

Putting the pieces together and collecting all factors, one finally gets the following result:

$$H_{\rm tr} = 2 \left[\tilde{\alpha}_0^i \alpha_0^i + \sum_{n=1}^{\infty} \left(\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \right) \right] \, .$$