LIE GROUPS EXERCISES

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Exercises connected to the lecture course Lie Groups 2006.

June 21, 2006

1. Lecture April 24

Consider matrices A with two rows and columns(2×2 -matrices), and only real coefficients. We demand that A be orthogonal($A\,\tilde{A} = \mathbb{I}$), and that $\det(A) = 1$.

a. Show that these matrices form a group, and that one parameter (de rotation angle) suffices to describe the matrix. This group is called $SO(2)$.

b. Show that these matrices commute: two matrices of this type, A and B, obey $[\,A,\,B\,] = 0$. Such groups are called *Abelian*.

2. Lecture April 24

Consider the definition of a *group* as in the notes. A group obeys the 4 axioms listed there.

a. Derive, only using these axioms, that the "left-inverse" R^{-1} defined at point 4, also serves as a "right-inverse" is: $R\,R^{-1} = \mathbb{I}$ (Hint: Consider $R^{-1}\,R\,R^{-1}$ and make use of the fact that also R^{-1} has an inverse).

b. Prove that the unit element is also the "right-unit-element": $R = R\,\mathbb{I}$ for every R (Hint: multiply at the right with $R^{-1}\,R$)

3. Lecture April 24

We give a few elements of a representation of the three-dimensional rotation group $SO(3)$ that consists of 2 dimensional matrices:

$$R_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \tag{3.1}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} , \tag{3.2}$$

Compute the matrices belonging to some other elements of this group, such as

$$R_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{3.3}$$

4. Lecture May 1

Let $R_{(1)}$ be a rotation over 90° around the z-axis and $R_{(2)}$ a rotation over 45° around the x-axis.

a. Write the corresponding matrices.

b. Compute the product matrix $R_{(3)} = R_{(2)}\, R_{(1)}$.

c. The matrix $R_{(3)}$ may also be viewed as a rotation over an angle φ around an axis $(a_1,\, a_2,\, a_3)$. Show that

$$\mathrm{Tr}\,\big(R_{(3)}\big) = 1 + 2\cos\varphi\,, \tag{4.1}$$

and compute the angle φ.

d. The coordinates of the axis can be deduced from the *antisymmetric part* of $R_{(3)}$, see equation (3.10) of the notes. Find the orientation of this axis.

e. Check that the length of the vector indeed produces the sinus of the rotation angle.

5. Lecture May 1

Consider the discussion of page 16—20 of the notes, but now applied to rotations in a *four* dimensional space. Thus, we have vectors $(x,\, y,\, z,\, u)$ that we rotate using orthogonal 4×4 matrices.

a. Show that there are 6 linearly independent generators. We may call them $L_1,\, \cdots,\, L_6$. Thus, a 4×4 matrix R can be written as

$$R = \exp\Big(i \sum_{k=1}^{6} \alpha_k\, L_k\Big) \tag{5.1}$$

b. Make a list of all structure constants $c_{ij}^{\,k}$.

6. Lecture May 1.

a. Derive equation (3.32) (see the notes) for the structure constants $c_{ij}^{\,m}$.

b. Show that, if we assume L_i to be hermitean, the structure constants are *purely imaginary.* Show furthermore that these constants are antisymmetric in the indices i en j: $c_{ij}^{\,m} = -c_{ji}^{\,m}$.

c. Now introduce the so-called *metric tensor* g_{ij} (using sommation convention):

$$g_{ij} \stackrel{\text{def}}{=} c_{ik}^{\ m} c_{mj}^{\ k} \ . \tag{6.1}$$

Show that g_{ij} ia symmetric: $g_{ij} = g_{ji}$.

d. Show (with the use of (3.32)) that the 'normalized structure constants', $\tilde{c}_{kij} = g_{km} c_{ij}^{\ m}$ are completely antisymmetric:

$$\tilde{c}_{kij} = \tilde{c}_{ijk} = -\tilde{c}_{kji} \ . \tag{6.2}$$

Often, we *normealize* the L_i in such a way that $g_{ij} = \delta_{ij}$.

7. Lecture May 1.

Study Appendix D: The Campbell-Baker-Hausdorff formula. It implies that, given two matrices A and B, a matrix C exists, such that

$$e^A \, e^B = e^C \ , \tag{7.1}$$

and that C can be written as a power series in A and B consisting *exclusively of commutators*, and no direct products.

a. Admitting complex numbers, show that there are several matric C_i such that (7.1) holds.
Hint: diagonalize C and add $2\pi i$ to one of its eigenvalues,

b. There are several ways to prove the theorem. Instead of equation (D.2) we can also define

$$e^{C(x)} = e^{xA} \, e^B \ . \tag{7.2}$$

Use the same method as in the Appendix to derive that

$$\frac{\mathrm{d}C(x)}{\mathrm{d}x} \quad = \quad \{A, \frac{C}{1-e^{-C}}\} \quad = \quad A + \tfrac{1}{2}[A,C] + \tfrac{1}{12}[[A,C],C] + \cdots \ . \tag{7.3}$$

c. Use this result to compute the coefficients in (D.29).

This exercise is a bit more difficult than what will be required at the test of this course. The proof of the CBH formula will not have to be reproduced in the test.

8. Lecture May 8

Consider again the group of orthogonal rotations in 4 dimensions. The elementary representation is formed by 4-vectors themselves. The matrices R en D then remain the same. This is the **4** representation. Write the components of the 4 vector as $x^\mu,\ \mu = 1, 2, 3$ or 4.

a. Now consider the 16-dimensional space spanned by the product of two such vectors. Write $A^{\mu\nu},\ \mu,\nu = 1,\cdots,4$. Now give the matrix D if under a rotation

$$A^{\mu\nu} \to R^{\mu\alpha} R^{\nu\beta} A^{\alpha\beta}\ . \tag{8.1}$$

b. This **16** representation however is not irreducible. Show that the two subspaces spanned by the *symmetric* tensors $A^{\mu\nu}$, for which $A^{\mu\nu} = A^{\nu\mu}$, and those of the *antisymmetric* tensors, $A^{\mu\nu} = -A^{\nu\mu}$, are invariant subspaces, with dimensions 10 and 6, respectively.

c. Show that the 10-dimensional representation can be split up further (by looking at the *trace* $A^{\mu\mu}$: $\mathbf{10} = \mathbf{9} \oplus \mathbf{1}$.

d. But also the antisymmetric representation **6** can be split up. Consider two kinds of tensors:

$$A_+^{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A_+^{\alpha\beta}\ , \tag{8.2}$$

$$A_-^{\mu\nu} = -\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} A_-^{\alpha\beta}\ , \tag{8.3}$$

to establish that there are two invariant subspaces: $\mathbf{6} = \mathbf{3}_L \oplus \mathbf{3}_R$. Later we will find out that the $\mathbf{3}_L$ en $\mathbf{3}_R$ are not unitarily equivalent.

9. Lecture May 8

Consider the generators L_i of the rotations as given in equation (3.14) of the notes.

a. They generate what we call the 'adjoint representation' of the group $SO(3)$. Find the eigenvalues $m = -1,\ 0,\ +1$ of L_3, and the associated eigenvectors (in the basis where (3.14) holds), which we will call $|m\rangle$.

b. Construct the matrices L_+ and L_-, and find out how they act on the states $|m\rangle$.

c. Check whether you indeed get $|m \pm 1\rangle$, with the norm factors $\sqrt{2 - m(m \pm 1)}$ as expected according to chapter 5.

d. Compute the Casimir operator $\sum_i L_i^2$ and check whether indeed the quantum number ℓ is 1 .

10. Lecture May 8 and May 15

This exercise is more difficult. It exhibits procedures that will be further elaborated on in the lectures.

A representation A of $SO(3)$ has basis elements ψ^A_α, $\alpha = 1, \cdots, N_A$, and a representation B has basis elements ψ^B_κ, $\kappa = 1, \cdots, N_B$. Consider the product representation $A \otimes B$ having the products $\psi^A_\alpha \psi^B_\kappa$ as its basis elements. The dimension of this representation is $N_A N_B$.

a. Show that the generators of this representation can be written as $I_i^{A\otimes B} = I_i^A + I_i^B$, where I_i^A only act on the indices α of representation A and I_i^B only on the indices κ of representation B.

b. Show that $[I_i^A,\, I_j^B] = 0$. We still have $[I_i^A,\, I_j^A] = i\varepsilon_{ijk} I_k^A$ and $[I_i^B,\, I_j^B] = i\varepsilon_{ijk} I_k^B$.

c. Show that the operators $I_i^{A\otimes B}$ obey the correct commutation rules. Omitting superscript $A \otimes B$: $[I_i,\, I_j] = i\varepsilon_{ijk} I_k$.

d. Write the basis elements of $A \otimes B$ as $|m^A, m^B\rangle$. How do the operators $I_\pm$ act on this basis?

e. The new representation $A \otimes B$ is not irreducible. Consider the values that $m^{A\otimes B}$ can have and use that to derive that the maximal value of $\ell^{A\otimes B}$ must be equal to $\ell^A + \ell^B$.

f. Take the case $\ell^A = \frac{5}{2}$ and $\ell^B = \frac{3}{2}$. Show that there is only one state with $m^{A\otimes B} = 4$. It hust have $\ell^{A\otimes B} = 4$ (convince yourself of this by letting I_+ act on this state). It is indicated as $|\ell = 4,\ m = 4\rangle$. How many basis basis elements do we have with $m^{A\otimes B} = 3$, with $m^{A\otimes B} = 2$, etc. ?

g. Now let I_- act on them. Construct the state $|\ell = 4,\ m = 3\rangle$. Show that it is equal to $\sqrt{\frac{5}{8}}\,|\frac{3}{2},\, \frac{3}{2}\rangle + \sqrt{\frac{3}{8}}\,|\frac{5}{2},\, \frac{1}{2}\rangle$. Why do we still have $\ell = 4$?

h. There is an other state with $m^{A\otimes B} = 3$ orthogonal to the one found above. Construct this state and show that it is $|\ell = 3,\ m = 3\rangle$.

i. This way we can find all states $|\ell,\, m\rangle$. Indicating the representations by their dimension $N = 2\ell + 1$, show that $\mathbf{4} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{5} \oplus \mathbf{7} \oplus \mathbf{9}$.

j. In general: $\ell^{A\otimes B} = |\ell^A - \ell^B|,\, |\ell^A - \ell^B| + 1,\, \cdots,\, \ell^A + \ell^B$. Show that the sums of the dimensions of these irreducible parts match:

$$\sum_{\ell=|\ell^A-\ell^B|}^{\ell^A+\ell^B} (2\ell + 1) = (2\ell^A + 1)(2\ell^B + 1)\ . \tag{10.1}$$

11. Lecture May 15.

a. Show that 2×2-matrices X of the form $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, where a an b are arbitrary complex numbers, form a group because they obey the axioms of page 8.

b. Show that this remains to be true if we impose on a en b the condition that $|a|^2 + |b|^2 = 1$.

c. Show that this condition corresponds to $\det(X) = 1$.

12. Lecture May 15.

We consider analytical functions f of two complex variables φ^1, φ^2. The operators L_i, $i = 1, 2, 3,$ are defined as in eq. (6.33):

$$L_i = -\tfrac{1}{2}(\tau_i)^\alpha{}_\beta \varphi^\beta \frac{\partial}{\partial \varphi^\alpha} \ . \tag{12.1}$$

Show that they obey the commutation rules of $SU(2)$ (and $SO(3)$):

$$[L_i,\, L_j] = i\,\varepsilon_{ijk} L_k \ . \tag{12.2}$$

Note the minus sign in the definition!

13. Lecture May 22.

Consider now an experiment as in Chapter 7, but now the scattering of a particle with spin 1 against a spherically symmetric target. We write the representation in the basis where L_z is diagonal, so

$$\psi^{+1} = f(\vec r)\begin{pmatrix}1\\0\\0\end{pmatrix} , \qquad \psi^{0} = f(\vec r)\begin{pmatrix}0\\1\\0\end{pmatrix} , \qquad \psi^{-1} = f(\vec r)\begin{pmatrix}0\\0\\1\end{pmatrix} . \tag{13.1}$$

Now define the scattering function $F(\theta)$ as the 3×3 matrix

$$\begin{pmatrix} f_{++}(\theta) & f_{+0}(\theta) & f_{+-}(\theta) \\ f_{0+}(\theta) & f_{00}(\theta) & f_{0-}(\theta) \\ f_{-+}(\theta) & f_{-0}(\theta) & f_{--}(\theta) \end{pmatrix} . \tag{13.2}$$

a. Construct the matrix that generates infinitesimal rotations around the x axis in this basis: $L_x = \frac{1}{2}(L_+ + L_-)$.

b. We find the rotation over a finite angle ξ, or $\exp(i\xi L_x)$, by applying (3.10):

$$\exp(i\xi L_x) = \mathbb{I} + (\cos\xi - 1)L_x^2 + i\sin\xi\, L_x \ . \tag{13.3}$$

Show that this follows from $L_x^{2n+1} = L_x$ and compute $\exp(i\xi L_x)$.

c. Find out how $F(\theta)$ transforms under this rotation.

d. What is the most general angle dependence to be expected for f_{++}, f_{00} en f_{--}?

e. Now compute the angular distribution for scattering experiments with particles having $S_z = 1, 0,$ en -1. Arguments using parity symmetry may be used to find further restrictions for the parameters in this result, which otherwise would be allowed to be anything.

f. So, how do we distinguish particles with spin $\frac{1}{2}$ from particles with spin 1?

14. Lecture May 22.

Just like the ordinary nucleons $N = (p, n)$, the N^* has isospin $\frac{1}{2}$. Its mass is sufficient to decay, just like the Δ particles, into a nucleon and a pion. The resulting states $|N, \pi\rangle$ now must have total isospin $\frac{1}{2}$. Of all states occurring in formula (8.6), we now search for those combinations that have $I^{\text{tot}} = \frac{1}{2}$ and $I_3 = +\frac{1}{2}$.

a. Find the two basis elements with $I_3 = +\frac{1}{2}$. One combination must have $I^{\text{tot}} = \frac{3}{2}$ and one combination, orthogonal to that, has $I^{\text{tot}} = \frac{1}{2}$. Why are these combinations orthogonal to one another?

b. Show that the state $I^{\text{tot}} = \frac{1}{2}$ must obey $I_+|\psi\rangle = 0$. Now find this state.

c. The state $I^{\text{tot}} = \frac{1}{2}$, $I_3 = -\frac{1}{2}$ is now obtained from the result of (b) by letting I_- act on it. Compute this state.

d. An alternative method is to solve the equation $I_-|\psi\rangle = 0$. Show that this produces the same result, apart from a possible phase factor (or minus sign).

e. Now compute the decay probabilities for

$$\begin{aligned} N^{*+} &\to p + \pi^0 \ , \qquad & N^{*+} &\to n + \pi^+ \ , \\ N^{*0} &\to p + \pi^- \ , \qquad & N^{*0} &\to n + \pi^0 \ . \end{aligned} \tag{14.1}$$

Observe the difference with the Δ decay, Eq. (8.14).

15. Lecture May 29.

The $K \to 2\pi$ decay. The K_S^0 particle can decay into two pions, and the ratio of the decay probabilities is found to be close to $(\pi^+\pi^-) : (2\pi^0) \approx 2 : 1$. We wonder which isospin state that is. Consider the 9 $|\pi, \pi\rangle$ states.

a. Using the operators I_+ en I_-, show that the $I^{\text{tot}} = 2,\ I_3 = m$ states, with $m = 2, 1, 0, -1, -2$, are:

$$\begin{aligned}
|2,\ 2\rangle &= |\pi^+,\ \pi^+\rangle \\
|2,\ 1\rangle &= \tfrac{1}{\sqrt{2}}\Big(|\pi^+,\ \pi^0\rangle + |\pi^0,\ \pi^+\rangle\Big) \\
|2,\ 0\rangle &= \tfrac{1}{\sqrt{6}}\Big(|\pi^+,\ \pi^-\rangle + 2|\pi^0,\ \pi^0\rangle + |\pi^-,\ \pi^+\rangle\Big) \\
|2, -1\rangle &= \tfrac{1}{\sqrt{2}}\Big(|\pi^0,\ \pi^-\rangle + |\pi^-,\ \pi^0\rangle\Big) \\
|2, -2\rangle &= |\pi^-,\ \pi^-\rangle\ ,
\end{aligned} \tag{15.1}$$

The $I^{\text{tot}} = 1,\ I_3 = m$ states are:

$$\begin{aligned}
|1,\ 1\rangle &= \tfrac{1}{\sqrt{2}}\Big(|\pi^+,\ \pi^0\rangle - |\pi^0,\ \pi^+\rangle\Big) \\
|1,\ 0\rangle &= \tfrac{1}{\sqrt{2}}\Big(|\pi^+,\ \pi^-\rangle - |\pi^-,\ \pi^+\rangle\Big) \\
|1, -1\rangle &= \tfrac{1}{\sqrt{2}}\Big(|\pi^0,\ \pi^-\rangle - |\pi^-,\ \pi^0\rangle\Big)\ ,
\end{aligned} \tag{15.2}$$

and the $I^{\text{tot}} = 0,\ I_3 = 0$ state is:

$$|0, 0\rangle = \tfrac{1}{\sqrt{3}}\Big(|\pi^+,\ \pi^-\rangle - |\pi^0,\ \pi^0\rangle + |\pi^-,\ \pi^+\rangle\Big) \tag{15.3}$$

b. Show that the observation is consistent with the suspicion that $I^{\text{tot}} = 0$.

c. The states with $I^{\text{tot}} = 1$ are in fact excluded because the (ordinary) spin of K_S^0 as well as that of the pions is zero (a fact that in reality had to established as well using arguments of the type described here). The pions are bosons, and therefore their states are spherically symmetric after the decay of the K particle. The minus sign is therefore not permitted. Show that the decay $K^+ \to \pi^+ + \pi^0$ therefore must terminate in an isospin 2 state. This decay turns out to have a much smaller amplitude than the K_S^0 decay. Thus one concludes that, although isospin in this (weak) decay is not conserved, nature has a strong preference for the decay towards the state with isospin 0. The (complicated) K physics will not be discussed further in these lectures.

16. Lecture June 12.

According to Kepler's laws, a planet moves along an elliptical orbit with the Sun in one of its two focal points. The semi-major axis of the ellipse is called a, the semi-minor axis is b, and the distance between a focal point and the center of the ellipse is called c. We have $a^2 = b^2 + c^2$. The fact that this elliptical orbit follows from Newton's laws is known from the lecture courses in classical mechanics. The same laws hold for a classical electron in its orbit around a classical proton. For that, we write the Hamiltonian (9.1). This is also the energy of the electron. One finds that the total energy in this ellipse is $-e^2/2a$.

a. Now derive that the total angular momentum L is equal to

$$L = e\,b\sqrt{\frac{\mu}{a}}\ . \tag{16.1}$$

To see this, choose one or more special points on the ellipse.

Check the calculation on page 47 where it is shown that the Runge-Lenz vector is conserved in time.

b. Again choose one or more special points in the orbit to compute the Runge-Lenz vector. Show that it points along the major axis of the ellipse, and that its length equals c/a. This is also the *eccentricity* ε of the ellipse.

17. Lecture June 12

Suppose that also in the quantum mechanical case formula (9.7) would be used for the definition of $\vec K$ (and not the symmetrized form (9.16)). Show that, in that case, $\vec K$ is no longer hermitean, and that the *anti*-hermitean part equals $\frac{1}{\mu e^2}\, i\,\hbar\,\vec p$. This is certainly not conserved in time.

18. June 12

It is not difficult to show that

$$\begin{aligned} [L_i,\, r_j] &= i\hbar\varepsilon_{ijk}r_k \\ \text{and} \qquad [L_i,\, p_j] &= i\hbar\varepsilon_{ijk}p_k\ . \end{aligned} \tag{18.1}$$

Now show that, if two arbitrary vectors $\vec A$ and $\vec B$ both obey the commutation rules with L_i as in (18.1), and if an other operator R commutes with all L_i, then the same commutation relations are obeyed by vectors $\vec C$ and $\vec D$ defined according to $\vec C = R\,\vec A$, and $\vec D = \vec A \times \vec B$, so

$$\begin{aligned} [L_i,\, C_j] &= i\hbar\varepsilon_{ijk}C_k \\ \text{and} \qquad [L_i,\, D_j] &= i\hbar\varepsilon_{ijk}D_k\ . \end{aligned} \tag{18.2}$$

This is the so-called Wigner-Eckhart theorem. With this, we prove both (9.19) and (9.22) in one stroke.

19. June 12

a. Check the calculations that lead to the equations $[\vec K,\, H] = 0$ (9.25) and $\vec K \cdot \vec L = 0$ (9.30) for the quantum mechanical Runge-Lenz vector (9.16).

b. Show that

$$(\vec{L}\times\vec{p})\cdot(\vec{p}\times\vec{L}) = -p^2L^2\ . \tag{19.1}$$

Make sure that no extra terms arise by correctly manipulating the commutation rules[1]

c. We now calculate $\vec{K}^2$ by writing $\vec{K}$ as

$$\vec{K} = \frac{1}{\mu e^2}(\vec{L}\times\vec{p} - \hbar i\vec{p}) + \frac{\vec{r}}{r} \;=\; \frac{1}{\mu e^2}(-\vec{p}\times\vec{L} + \hbar i\vec{p}) + \frac{\vec{r}}{r}\ , \tag{19.2}$$

after which we multiply one expression with the other, making use of b. Now prove (9.31).

20. June 12

Consider the quantum states of the hydrogen atom at given n.

a. Show that both L_3^+ and L_3^- take the values $-\frac{1}{2}(n-1),\ -\frac{1}{2}(n-1)+1,\ \cdots,\ \frac{1}{2}(n-1)$..

b. What are now the possible values of $L_3 = L_3^+ + L_3^-$, and what is the degree of degeneracy at these values?

c. Use this to derive that ℓ takes the integral values 0, 1, $\cdots$ $n-1$ all exactly once, according to equation (9.46).

21. June 19

The fundamental representation of $SU(3)$ corresponds to the three quarks,

$$\phi^\alpha \;=\; \begin{pmatrix} u \\ d \\ s \end{pmatrix}\ , \tag{21.1}$$

of which the first two form a doublet under isospin-$SU(2)$. The strangeness of u en d is 0, that of s is -1.

a. Indicate which pattern these quarks form if we plot isospin against strangeness, as is also done in Figures 5 and 6.

b. Assuming that Δ^- consists of three d quarks, Δ^{++} of three u quarks and Ω^- of three s quarks, calculate the electric charge of these quarks, and compare the diagram obtained with Figure 6.

[1] In the lecture the other combination was computed, which does produce extra terms.

c. The **6** representation is given by the symmetric tensors $\phi^{\alpha\beta}$. Show that

$$\mathbf{6}_{\rm SU(3)} = (\mathbf{3}+\mathbf{2}+\mathbf{1})_{,\rm SU(2)} . \tag{21.2}$$

d. Show that this is an exotic representation , and give the position of these particles in the I_3 - S diagram.

22. June 19

Product representations in $SU(3)$. We wish to know how the product representation $\mathbf{8}\otimes\mathbf{8}$ splits into irreducible representations of $SU(3)$. Consider only the antisymmetric part, $(\mathbf{8}\otimes\mathbf{8})_a$:

$$X^{ab} = -X^{ba} \ , \qquad a,\, b = 1,\, \cdots,\, 8 \ . \tag{22.1}$$

a. Show that this representation is 28 dimensional.

b. Show that the representation $\Big((\mathbf{8}+\mathbf{1})\otimes(\mathbf{8}+\mathbf{1})\Big)_a$ can be written as a tensor $Y^{\alpha\beta}_{\gamma\delta} = -Y^{\beta\alpha}_{\delta\gamma}$, where the indices α,β,γ and δ take all values $1,\cdots,3$.

c. Show that this representation splits into two pieces, each with 18 dimensions:

$$\begin{aligned} Y^{\alpha\beta}_{\gamma\delta} &= Y^{+\,\alpha\beta}_{\ \gamma\delta} + Y^{-\,\alpha\beta}_{\ \gamma\delta} \ ; \\ Y^{+\,\alpha\beta}_{\ \gamma\delta} &= Y^{+\,\beta\alpha}_{\ \gamma\delta} \ = \ -Y^{+\,\alpha\beta}_{\ \delta\gamma} \ , \\ \text{and} \qquad Y^{-\,\alpha\beta}_{\ \gamma\delta} &= -Y^{-\,\beta\alpha}_{\ \gamma\delta} \ = \ Y^{+\,\alpha\beta}_{\ \delta\gamma} \ . \end{aligned} \tag{22.2}$$

d. Now use $\varepsilon^{\gamma\delta\kappa}$ to raise all indices of Y^+. We then have three indices. Show that this tensor splits into a completely symmetric tensor and the rest. So we have that Y^+ splits into a **10** and an **8**.

e. Show that Y^- splits into an **8** and a $\overline{\mathbf{10}}$.

f. Conclude that

$$\Big((\mathbf{8}+\mathbf{1})\otimes(\mathbf{8}+\mathbf{1})\Big)_a = \mathbf{10}+\mathbf{8}+\mathbf{8}+\overline{\mathbf{10}} \ . \tag{22.3}$$

g. Now derive that

$$(\mathbf{8}\otimes\mathbf{8})_a = \mathbf{10}+\mathbf{8}+\overline{\mathbf{10}} \ . \tag{22.4}$$