String Theory Exercises*

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Exercise 1

Let us consider the action describing a point particle of mass m moving freely in a $d+1$ dimensional Minkowski space–time. It will be expressed as the invariant length of the world–line

$$S = \int_{\tau_0}^{\tau_1} d\tau\,\mathcal{L} = -m\int_{\tau_0}^{\tau_1} ds = -m\int_{\tau_0}^{\tau_1} d\tau\,\sqrt{-\dot{x}^\mu \dot{x}_\mu}. \tag{1.1}$$

where $x^\mu(\tau)$ is the space–time position of the particle at the proper time τ and $\dot{x}^\mu = dx^\mu/d\tau$. The space–time metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,\ldots,+1)$.

- Show that S is invariant with respect to *reparametrization* of the world line: $\tau \to \tau' = \tau'(\tau)$.
- Compute the momentum of the particle $p^\mu = \eta^{\mu\nu}\,\delta\mathcal{L}/\delta\dot{x}^\nu$ and show that it describes a particle of mass m.
- Find the equation of motion for $x^\mu(\tau)$ by minimizing the action with respect to a variation $x^\mu \to x^\mu + \delta x^\mu$ and find the most general solution. In particular show that, if we give a physical interpretation to τ as being the time, namely if we set $x^0(\tau) \propto \tau$, the solution has $v^\mu = \dot{x}^\mu = \mathrm{const}$. Show that on the solutions the action can be written in the form:

$$S = -m\,|x_1 - x_0|. \tag{1.2}$$

$$\begin{aligned}
|x_1 - x_0| &= \sqrt{-(x_1^\mu - x_0^\mu)(x_{1\mu} - x_{0\mu})} \\
x_0^\mu &= x^\mu(\tau_0)\ ;\ \ x_1^\mu = x^\mu(\tau_1) \\
x_1^\mu - x_0^\mu &= v^\mu\,(\tau_1 - \tau_0).
\end{aligned} \tag{1.3}$$

*References to equations in the lecture notes refer to the 10/02/04 version of the notes.

and that the momentum can be written as:

$$p^\mu = m \frac{(x_1^\mu - x_0^\mu)}{|x_1 - x_0|}. \tag{1.4}$$

(for later convenience we allow p^0 to have also negative values and we interpret the energy of the particle to be $E = |p^0|$. According to this notation, in a scattering process, the incoming particles will have $p^0 = E > 0$ while the outgoing will have $p^0 = -E < 0$.)

- Now consider a different action on the same world line of the particle, but this time function of two independent quantities, namely $e(\tau)$ and $x^\mu(\tau)$:

$$S' = \frac{1}{2} \int d\tau \, (e^{-1} \, \dot{x}^\mu \dot{x}_\mu - e m^2) \tag{1.5}$$

Show that S' is invariant with respect to the reparametrization transformation in τ which, in its infinitesimal form is:

$$\begin{aligned} \delta x^\mu &= \epsilon(\tau) \, \dot{x}^\mu \\ \delta e &= \frac{d}{d\tau} (\epsilon(\tau) \, e) \\ \epsilon(\tau_0) &= \epsilon(\tau_1) = 0. \end{aligned} \tag{1.6}$$

[**Hint:** *To show invariance of the action under the above infinitesimal transformations it suffices to show that the corresponding variation of the action is a total derivative of the form written below:*

$$\mathcal{L}(e + \delta e, \dot{x}^\mu + \delta \dot{x}^\mu) = \mathcal{L}(e, \dot{x}^\mu) + \frac{d}{d\tau} (\epsilon \mathcal{L}) \, . \tag{1.7}$$

for these transformations the above result follows without the use of the equations of motion. Since by hypothesis ϵ vanishes at the extrema of integration the total derivative in eq. (1.7) does not contribute.]

Compute the field equations from S' corresponding to the fields $e(\tau)$ and $x^\mu(\tau)$ and show that S' is classically equivalent to S.

- Consider a scattering process of N particles described by the world lines of the particles which start at different points x_i^μ $(i = 1, \ldots, N)$ and intersect in the same scattering point y^μ. Let the process be described by the total action $S = \sum_{i=1}^N S_i$, S_i being the world line actions of the single particles, by minimizing S with respect to the position y^μ of the scattering point deduce the momentum conservation condition for the process: $\sum_{i=1}^N p_i^\mu = 0$ (recall that in our notation the incoming particles have $p^0 > 0$ and the outgoing have $p^0 < 0$). [**Hint:** *Use eq. (1.2) to express the single actions as $S_i = -m_i \, |y - x_i|$. Then from the condition $\delta S / \delta y^\mu = 0$ and the expression (1.4) for the momenta deduce* $\sum_{i=1}^N p_i^\mu = 0$.]

Exercise 2

A generic surface $\mathcal{S}$ spanned by affine parameters τ, σ and embedded in a higher dimensional space (ambient space) with coordinates X^μ, can be described by the parametric equations $X^\mu = X^\mu(\tau,\, \sigma)$. An infinitesimal element of its surface $d\mathcal{S}$ can be represented by a tensor $d\mathcal{S}^{\mu\nu}(\tau,\, \sigma) = (dX^\mu \wedge dX^\nu)_{|\mathcal{S}} = \Sigma^{\mu\nu}(\tau,\, \sigma) d\tau d\sigma$ which defines the projection of $d\mathcal{S}$ on the plane (X^μ, X^ν), the tensor $\Sigma^{\mu\nu}(\tau,\, \sigma)$ defines the plane tangent to the surface $\mathcal{S}$ at the point $(\tau,\, \sigma)$ and is defined as follows:

$$\Sigma^{\mu\nu} = \partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu \,.$$

The area of the surface $A(\mathcal{S})$ is then defined by the integral:

$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{\frac{1}{2}\, d\mathcal{S}_{\mu\nu} d\mathcal{S}^{\mu\nu}} = \int_{\mathcal{S}} d\tau d\sigma \sqrt{\frac{1}{2}\, \Sigma_{\mu\nu}\Sigma^{\mu\nu}} \tag{2.1}$$

Compute the Σ tensor and the area of the surface given by:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= -B\tau \\ X^2 &= \sigma \end{aligned}$$

where A and B are constants, $\tau \in (0,1)$ and $\sigma \in (0,1)$. You may consider the metric of the ambient space to be Euclidean. Compare the result of the integral with what you would expect.

Exercise 3

Consider the Nambu–Goto action described by equation (2.9) of the lecture notes. Show that the expression in the square root can be written as the determinant of a 2×2 matrix (induced metric) $h_{\alpha\beta}$ defined as

$$h_{\alpha\beta} \;=\; \partial_\alpha X^\mu \, \partial_\beta X_\mu \tag{3.1}$$

In the Nambu–Goto action the only independent function is $X^\mu(\sigma, \tau)$. It is possible to reformulate the theory in a classically equivalent way using the Polyakov action which describes $h_{\alpha\beta}(\sigma, \tau)$ and $X^\mu(\sigma, \tau)$ as independent fields and has the advantage of not having the square root in the integral (see next exercise)

Exercise 4

Consider the Polyakov action of a string moving on a D–dimensional Minkowski background (with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$):

$$S = -\frac{T}{2}\int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, h^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu\, \partial_\beta X_\mu \tag{4.1}$$

$$\begin{aligned}
\sigma^\alpha &= \{\sigma, \tau\} \ ; \ X^\mu = X^\mu(\sigma^\alpha)\, ; \, \sigma \in [0, \pi]\, ; \, \tau \in (-\infty, \infty) \\
\Sigma &= \text{world sheet} \ ; \ h_{\alpha\beta}(\sigma^\gamma) = \text{metric on } \Sigma \\
h(\sigma^\gamma) &= -\det(h_{\alpha\beta}) \ ; \ \alpha = 1, 2\, ; \, \mu = 0, \ldots, \mathrm{D}-1
\end{aligned}$$

where we have used the following short hand notation for partial derivatives: $\partial_\gamma = \frac{\partial}{\partial \sigma^\gamma}$. Moreover whenever repeated upper and lower indices occur summation is understood: $v^\alpha w_\alpha = \sum_\alpha v^\alpha w_\alpha$.

- Show that the *local*, i.e. σ^α–dependent, symmetry transformations are:

$$\underline{\text{reparametrization}}: \qquad \sigma^\alpha \to \sigma^{\alpha\prime} = \sigma^{\alpha\prime}(\sigma) \tag{4.2}$$

$$\underline{\text{Weyl transformations}}: \qquad h_{\alpha\beta} \to \Omega^2(\sigma^\alpha)\, h_{\alpha\beta} \tag{4.3}$$

- Compute the energy momentum tensor $T_{\alpha\beta}$ defined by:

$$\begin{aligned}
h_{\alpha\beta} &\to h_{\alpha\beta} + \delta h_{\alpha\beta} \Rightarrow S \to S + \delta S \\
\delta S &= -\frac{T}{2}\int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, \delta h_{\alpha\beta}\, T^{\alpha\beta}
\end{aligned} \tag{4.4}$$

  which condition on $T_{\alpha\beta}$ does invariance under Weyl transformations imply?

- Show that the *global*, i.e. σ^α–independent, transformations on the X^μ fields which leave S invariant are the Poincaré transformations:

$$\begin{aligned}
X^\mu &\to X'^\mu = \Lambda^\mu{}_\nu\, X^\nu + a^\mu \\
\eta_{\mu\nu}\, \Lambda^\nu{}_\rho\, \Lambda^\mu{}_\gamma &= \eta_{\rho\gamma}
\end{aligned} \tag{4.5}$$

  where both the Lorentz transformation Λ and the translation parameter a^μ do not depend on σ^α.

- Write the equations of motion for the fields $h_{\alpha\beta}(\sigma)$ and $X^\mu(\sigma)$ and show that S is *classically* equivalent to the Nambu–Goto action S':

$$S' = -T\int_\Sigma d\sigma^2 \sqrt{-\det\left(\partial_\alpha X^\mu\, \partial_\beta X_\mu\right)} \tag{4.6}$$

Exercise 5

By fixing reparametrization invariance let us reduce the world sheet metric to the form:

$$h_{\alpha\beta}(\sigma) \;=\; \lambda(\sigma)\,\eta_{\alpha\beta} \;=\; \lambda(\sigma)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \tag{5.1}$$

- write the action S with this metric
- consider an infinitesimal coordinate transformation on the world sheet:

$$\sigma^\alpha \to \sigma^{\alpha\prime} \;=\; \sigma^\alpha + \epsilon^\alpha(\sigma) \tag{5.2}$$

  which implies that we transform X^μ as follows:

$$X^\mu \to X^\mu + \delta X^\mu \;=\; X^\mu + \epsilon^\gamma\,\partial_\gamma X^\mu$$

  As it can be deduced from eq. (3.1), the metric $h_{\alpha\beta}$ will transform as follows:

$$\delta h_{\alpha\beta} \;=\; (\partial_\alpha\epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta\epsilon^\gamma)\,h_{\gamma\alpha} + \epsilon^\gamma\partial_\gamma h_{\alpha\beta} \tag{5.3}$$

  After fixing the metric to the form (5.1) there is still a residual invariance of the action under *conformal* transformations. A conformal transformation is defined as a coordinate transformation whose only effect is to rescale the metric, namely such that the corresponding infinitesimal variation of the metric has the general form:

$$\delta h_{\alpha\beta} \;=\; C(\sigma)\,h_{\alpha\beta} + \epsilon^\gamma\partial_\gamma h_{\alpha\beta} \tag{5.4}$$

  where $C(\sigma)$ is an infinitesimal function of σ^α which depends on the infinitesimal coordinate shift ϵ. Using equ. (5.3) show that the conformal transformations are generated by an infinitesimal parameter ϵ fulfilling the following condition:

$$(\partial_\alpha\epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta\epsilon^\gamma)\,h_{\gamma\alpha} \;=\; (\partial_\gamma\,\epsilon^\gamma)\;h_{\alpha\beta} \tag{5.5}$$

  recall the convention on repeated indices and partial derivation: $(\partial_\gamma\,\epsilon^\gamma = \sum_\gamma \frac{\partial}{\partial\sigma^\gamma}\,\epsilon^\gamma)$ and find the expression of $C(\sigma)$ in equ. (5.4) in terms of ϵ.
- using the light–cone coordinates:

$$\sigma^\pm \;=\; \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{5.6}$$

  show that conformal transformations are characterized by $\epsilon^+ = \epsilon^+(\sigma^+)$ and $\epsilon^- = \epsilon^-(\sigma^-)$.
- Write $T_{\alpha\beta}$ and the conditions on it due to energy–momentum conservation and Weyl invariance, in light–cone coordinates.

Exercise 6

Consider the global symmetries of S, i.e. the Poincaré transformations in their infinitesimal form:

$$
\begin{aligned}
X^\mu &\rightarrow X^\mu + \delta X^\mu && (6.1)\\
\delta X^\mu &= a^\mu \qquad \text{(translations)} && (6.2)\\
\delta X^\rho &= \omega_{\mu\nu}\,(M^{\mu\nu})^\rho{}_\sigma\, X^\sigma \quad \text{(Lorentz)} && (6.3)\\
(M^{\mu\nu})^\rho{}_\sigma &= \eta^{\nu\rho}\,\delta^\mu_\sigma - \eta^{\mu\rho}\,\delta^\nu_\sigma &&\\
\eta^{\mu\nu} &= \mathrm{diag}(-1,+1,\ldots,+1) && (6.4)
\end{aligned}
$$

where a^μ and $\omega_{\mu\nu}$ are σ^α–independent.

- Compute the corresponding conserved Noether currents J^μ_α and $J^{\mu\nu}_\alpha$ by computing the variation of the Polyakov action S with respect to the transformations (6.2) and (6.3) and expressing it (using the field equations) in the form:

$$
\delta S = \int d\sigma d\tau\, \partial^\alpha \left(J^\mu_\alpha\, a_\mu + J^{\mu\nu}_\alpha\, \omega_{\mu\nu}\right) \qquad (6.5)
$$

  recall that we are using here and in all the following exercises $h_{\alpha\beta} = \eta_{\alpha\beta}$

- Write the equations of motion for X^μ and solve them with the following boundary conditions:

$$
\begin{aligned}
&\underline{\text{Neumann:}} && \partial_\sigma X^\mu(\tau,\sigma=0) = \partial_\sigma X^\mu(\tau,\sigma=\pi) = 0\\
&\underline{\text{Closed string:}} && X^\mu(\tau,\sigma) \equiv X^\mu(\tau,\sigma+\pi) \quad \forall\sigma
\end{aligned}
$$

  showing that the most general solution will have the form:

$$
X^\mu(\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad (6.6)
$$

  (express solution through Fourier mode expansion and impose the boundary conditions as constraints on the coefficients)

- Show that for the above solutions the CM momentum:

$$
P^\mu = \int_0^\pi d\sigma\, J^\mu_\tau \qquad (6.7)
$$

  is conserved.

- Write the expression of P^μ for the above solutions.

Exercise 7

Consider an open string satisfying the usual Neumann boundary conditions along the directions $X^0, \dots X^{D-2}$, but a different one on X^{D-1}. For the following two cases, compute the mode expansion of X^{D-1} as in the previous exercise.

- Dirichlet boundary conditions at both endpoints (DD):

$$\partial_\tau X^{D-1}(\tau, \sigma = 0) = 0 \,, \qquad \partial_\tau X^{D-1}(\tau, \sigma = \pi) = 0 \,. \tag{7.1}$$

  This has the interpretation of an open string with both ends on a D-brane, where "D" stands for Dirichlet. Is the momentum P^{D-1} along this direction conserved?

- Dirichlet boundary condition at one endpoint and Neumann at the other endpoint (ND):

$$\partial_\tau X^{D-1}(\tau, \sigma = 0) = 0 \,, \qquad \partial_\sigma X^{D-1}(\tau, \sigma = \pi) = 0 \,. \tag{7.2}$$

  This is an open string with one end on a D-brane and one free.

  [**Hint:** *Will the frequencies of the Fourier modes still be integer?*]

Exercise 8

Show that the following functions:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\sin(\tau)\cos(\sigma) \\ X^{i>2} &= 0 \end{aligned} \tag{8.1}$$

define a solution of the string field equations with the Neumann boundary conditions (i.e. describe a free open string). In particular show that it can be written in the form (6.6).

- Compute the energy $E = P^0$ and the angular momentum $\boldsymbol{J}$ for this solution.
- Show that the non linear constraints $T_{\alpha\beta} = 0$ are fulfilled as well, namely that:

$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0 \,, \qquad \partial_\tau X^\mu \, \partial_\sigma X_\mu = 0 \tag{8.2}$$

- Show that:

$$\frac{E^2}{|\boldsymbol{J}|} = \text{const.} = \frac{1}{\alpha'} \tag{8.3}$$

  and show that $T = 1/(2\pi\alpha')$

- Show that this solution describes an open string with the end points rotating at the speed of light.

Exercise 9

Consider the following parametric equations:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\cos(\tau)\sin(\sigma) \end{aligned} \tag{9.1}$$

Show that they describe a string solution, i.e. that one can write it in the form (6.6) and that the non linear constraints in eq. (8.2) are fulfilled. Which boundary conditions does the solution fulfill in the various space directions?

- Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to 2π with steps of $\pi/4$.
- Compute the conserved current J^μ_α associated with global translational invariance in space–time (see problem 6). Show that the component $P^{\mu=2}$ of the momentum defined by eq. (6.7) is not conserved.
- Compute the variation of the momentum between $\tau = 0$ and $\tau = \tau_0$ and prove the following relation:

$$P^{\mu=2}_{|\tau=\tau_0} - P^{\mu=2}_{|\tau=0} = \int_0^{\tau_0} d\tau \, \left(J^{\mu=2}_\sigma(\tau,\pi) - J^{\mu=2}_\sigma(\tau,0) \right) \tag{9.2}$$

  the above equation derives from the momentum conservation condition $\partial^\alpha J^\mu_\alpha = 0$ and the right hand side can be interpreted as the momentum flow in the direction $\mu = 2$ across the end–points of the string.

Exercise 10

Consider the following parametric equations:

$$\begin{aligned} X^0 &= 2A\tau \\ X^1 &= A\cos 2\tau \cos 2\sigma \\ X^2 &= A\sin 2\tau \cos 2\sigma \end{aligned} \tag{10.1}$$

Show that they describe a closed string solution and that one can write it in the form (6.6) and that it fulfills (8.2). Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to π with steps of $\pi/8$.

Now consider the following parametric equations:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(2\tau)\sin(2\sigma) \\ X^2 &= A\sin(4\tau)\sin(4\sigma) \end{aligned} \tag{10.2}$$

Show that although they can be written in the form (6.6) they do not fulfill the constraint (8.2).

Exercise 11

Write the components T_{++} and T_{--} of the energy momentum tensor in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions. In particular in the two cases find the corresponding expressions of the coefficients L_n and $\tilde{L}_n$ defined as follows:

$$\begin{aligned}
\underline{\text{open string:}} \qquad T_{++} &= \frac{\ell^2}{2} \sum_{n=-\infty}^{\infty} L_n \, e^{-in\sigma^+} \\
\underline{\text{closed string:}} \qquad T_{++} &= 2\ell^2 \sum_{n=-\infty}^{\infty} \tilde{L}_n \, e^{-2in\sigma^+} \\
T_{--} &= 2\ell^2 \sum_{n=-\infty}^{\infty} L_n \, e^{-2in\sigma^-}
\end{aligned} \tag{11.1}$$

The constraints $T_{\alpha\beta} = 0$ can be restated as $L_n = \tilde{L}_n = 0$ for all n. From the conditions $L_0 = \tilde{L}_0 = 0$ derive an expression for the mass squared $M^2 = -p^\mu p_\mu$ in terms of the coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ in both the open and closed string cases.

Exercise 12

Compute in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions the Hamiltonian associated with the Polyakov action:

$$H(\tau) \;=\; \int d\sigma \left(\dot{X}^\mu \, \frac{dP_\mu}{d\sigma} \right) - L(\tau) \tag{12.1}$$

where $\frac{dP_\mu}{d\sigma}$ and $L(\tau)$ are defined in eqs. (3.43) and (4.1) of the lecture notes. What is the relation between H and L_0, $\tilde{L}_0$ in the open and closed string cases?

Exercise 13

Let us restrict ourselves to the quantized open string case in the light-cone gauge. Derive the commutation relations between x^i, p^i and α_n^i from equations (4.5) and (4.3) of the lecture notes. Let us define the vacuum state $|p^\mu, 0\rangle$ such that $\alpha_n^i |p^\mu, 0\rangle = 0$ for $n > 0$ and $\alpha_0^i |p^\mu, 0\rangle = p^i |p^\mu, 0\rangle$. A generic open string state is obtained by applying the creation operators α_n^i $(n < 0)$ a finite number of times to $|p^\mu, 0\rangle$. Consider the operator N defined as follows:

$$N \;=\; \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i \tag{13.1}$$

Show that on the following state:

$$|p^\mu,\ N_{i,n}\rangle \quad = \quad \overbrace{\alpha^i_{-n}\dots\alpha^i_{-n}}^{N_{i,n}}\,|p^\mu,0\rangle \tag{13.2}$$

the operator N has the following value:

$$N\,|p^\mu,\ N_{i,n}\rangle \quad = \quad nN_{i,n}|p^\mu,\ N_{i,n}\rangle \tag{13.3}$$

Exercise 14

A solution of the string equations of motion can be written in the form (6.6). For the open string we have the further constraint that $X^\mu_L(\tau) = X^\mu_R(\tau)$ and that $X^\mu_L(\sigma^+)$ and $X^\mu_R(\sigma^-)$ are respectively periodic in σ^+ and σ^- of period 2π a part for a constant shift while for the closed string X^μ_L and X^μ_R need only be periodic in $\sigma \in (0,\pi)$ at fixed τ. The theory is also invariant under conformal transformations which can be written in the following infinitesimal form:

$$\begin{aligned}\sigma^+ &\quad\rightarrow\quad \sigma^+ + \epsilon^+(\sigma^+)\\ \sigma^- &\quad\rightarrow\quad \sigma^- + \epsilon^-(\sigma^-)\end{aligned} \tag{14.1}$$

where $\epsilon^+(\sigma^+)$ and $\epsilon^-(\sigma^-)$ are periodic in σ^+ and σ^- of period 2π. For the open string we further require that $\epsilon^+ = \epsilon^-$. In this exercise we will compute the infinitesimal generators of conformal transformations and their algebraic properties. Being periodic of period 2π in their argument, the infinitesimal functions can be expanded in Fourier series:

$$\epsilon^\pm(\sigma^\pm) \quad = \quad i\sum_{n=-\infty}^{+\infty}\epsilon^\pm_n e^{in\sigma^\pm} \tag{14.2}$$

where the reality condition implies that $\epsilon^\pm_{-n} = -(\epsilon^\pm_n)^\star$.
A generic scalar function $Y(\sigma^\pm)$ will transform under (14.1) as follows:

$$\begin{aligned}Y(\sigma^\pm) &\quad\rightarrow\quad Y(\sigma^\pm+\epsilon^\pm) \sim Y(\sigma^\pm) + \delta Y(\sigma^\pm)\\ \delta Y(\sigma^\pm) &\quad=\quad \epsilon^\pm\partial_\pm Y(\sigma^\pm) = i\sum_{n=-\infty}^{+\infty}\epsilon^\pm_n e^{in\sigma^\pm}\partial_\pm Y(\sigma^\pm)\end{aligned} \tag{14.3}$$

where we have used eq. (14.2) to derive the last expression. Let us define the following differential operators $\mathbf{L}^{(\pm)}_n$ on functions of $\sigma^\pm$ respectively:

$$\begin{aligned}\mathbf{L}^{(+)}_n\left(Y(\sigma^+)\right) &\quad=\quad ie^{in\sigma^+}\partial_+Y(\sigma^+)\\ \mathbf{L}^{(-)}_n\left(Y(\sigma^-)\right) &\quad=\quad ie^{in\sigma^-}\partial_-Y(\sigma^-)\\ \mathbf{L}^{(+)}_n\left(Y(\sigma^-)\right) &\quad=\quad 0\\ \mathbf{L}^{(-)}_n\left(Y(\sigma^+)\right) &\quad=\quad 0\end{aligned} \tag{14.4}$$

the product of two of these operators is defined by their consecutive action on a same function Y: $\mathbf{L}_n \mathbf{L}_m (Y) \equiv \mathbf{L}_n (\mathbf{L}_m (Y))$. The operators $\mathbf{L}_n^{(\pm)}$ are the generators of the infinitesimal conformal transformations (14.3), indeed we can write:

$$\delta Y(\sigma^\pm) = \sum_{n=-\infty}^{+\infty} \epsilon_n^\pm \mathbf{L}_n^{(\pm)} (Y) \tag{14.5}$$

- Show that taking $Y(\sigma^\pm) = \sigma^\pm$ and considering the definitions (14.4) the infinitesimal conformal transformation (14.1) follows from (14.5).
- Show that on a generic function $Y(\sigma^\pm)$ the following relations hold:

$$\begin{aligned} \left(\mathbf{L}_n^{(+)} \mathbf{L}_m^{(+)} - \mathbf{L}_m^{(+)} \mathbf{L}_n^{(+)}\right) \left(Y(\sigma^+)\right) &= (n-m)\mathbf{L}_{n+m}^{(+)} \left(Y(\sigma^+)\right) \\ \left(\mathbf{L}_n^{(-)} \mathbf{L}_m^{(-)} - \mathbf{L}_m^{(-)} \mathbf{L}_n^{(-)}\right) \left(Y(\sigma^-)\right) &= (n-m)\mathbf{L}_{n+m}^{(-)} \left(Y(\sigma^-)\right) \\ \left(\mathbf{L}_n^{(+)} \mathbf{L}_m^{(-)} - \mathbf{L}_m^{(-)} \mathbf{L}_n^{(+)}\right) \left(Y(\sigma^\pm)\right) &= 0 \end{aligned} \tag{14.6}$$

- Consider the following two constant infinitesimal transformations:

$$\begin{aligned} \delta\tau &= c \;;\; \delta\sigma = 0 \quad \text{time shift} \\ \delta\tau &= 0 \;;\; \delta\sigma = c \quad \text{world sheet rotation} \end{aligned} \tag{14.7}$$

  Show that they are generated respectively by the following differential operators:

$$\begin{aligned} \mathcal{O}_t &= -i\, c\, (\mathbf{L}_0^{(+)} + \mathbf{L}_0^{(-)}) \\ \mathcal{O}_s &= -i\, c\, (\mathbf{L}_0^{(+)} - \mathbf{L}_0^{(-)}) \end{aligned} \tag{14.8}$$

  namely that for the time shift $\delta\tau = \mathcal{O}_t(\tau) = c$ and $\delta\sigma = \mathcal{O}_s(\sigma) = 0$ while for the world sheet rotation $\delta\tau = \mathcal{O}_s(\tau) = 0$ and $\delta\sigma = \mathcal{O}_s(\sigma) = c$

Exercise 15

Let us define the following operators on the closed string states:

$$\begin{aligned} \mathcal{L}_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} : \alpha_m^i \alpha_{n-m}^i : \\ \tilde{\mathcal{L}}_n &= \frac{1}{2} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} : \tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i : \end{aligned} \tag{15.1}$$

where as usual α_n and $\tilde{\alpha}_n$ denote the right and left moving mode operators. Show that they fulfill the following commutation relations:

$$\begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + \frac{D-2}{12} n(n^2-1)\delta_{n+m} \\ \left[\tilde{\mathcal{L}}_n, \tilde{\mathcal{L}}_m\right] &= (n-m)\tilde{\mathcal{L}}_{n+m} + \frac{D-2}{12} n(n^2-1)\delta_{n+m} \\ \left[\mathcal{L}_n, \tilde{\mathcal{L}}_m\right] &= 0 \end{aligned} \tag{15.2}$$

these relations, apart from the term $\frac{D-2}{12}\, n(n^2-1)\delta_{n+m}$ on the left hand sides, are analogous to the relations (14.6) characterizing the generators of conformal transformations (Virasoro algebra).

Consider now the quantum version of the solution to the constraints in the light-come gauge, namely:

open string:

$$\alpha_n^- = \frac{1}{p^+\ell}\left(\mathcal{L}_n - a\,\delta_n\right)\ ;\ \alpha_n^+ = 0\ (n\neq 0)$$

closed string:

$$\begin{aligned}\alpha_n^- &= \frac{2}{p^+\ell}\left(\mathcal{L}_n - a\,\delta_n\right)\ ;\ \alpha_n^+ = 0\ (n\neq 0)\\ \tilde{\alpha}_n^- &= \frac{2}{p^+\ell}\left(\tilde{\mathcal{L}}_n - a\,\delta_n\right)\ ;\ \tilde{\alpha}_n^+ = 0\ (n\neq 0)\end{aligned} \tag{15.3}$$

Recalling that in the open string case $\alpha_0^\mu = \ell p^\mu$ while in the closed string case $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \ell p^\mu/2$, express in both cases $M^2 = -p^\mu p_\mu$ as function of α_n^i, $\tilde{\alpha}_n^i$ and a and in terms of the operator N defined for the open string case in equation (13.1) and for the closed string case by:

$$N = \sum_{n=1}^{\infty}\sum_{i=1}^{D-2}\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) \tag{15.4}$$

What is the lowest value of M^2 in both cases? Show that in the case $a=1$ these states are tachyons.

Derive in the open string case all the commutation relations between x^μ, p^μ and α_n^μ represented in the table in section 4.3 of the lecture notes.

Consider now the following quantum version of the expressions found in exercise 11 for the coefficients L_n and $\tilde{L}_n$ of the Fourier expansion of $T_{\pm\pm}$ in terms of the coefficients α_n and $\tilde{\alpha}_n$:

$$\begin{aligned}L_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_m^\mu\alpha_{n-m\,\mu}:\\ \tilde{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\tilde{\alpha}_m^\mu\tilde{\alpha}_{n-m\,\mu}:\end{aligned} \tag{15.5}$$

Using equations (15.3) show that in the light-cone gauge $L_n = \tilde{L}_n = a\,\delta_n$.

Consider the open string case in which $\mathcal{L}_n = \tilde{\mathcal{L}}_n$. Show that the following relations hold:

$$\left[\alpha_n^-,\,\alpha_m^-\right] = (n-m)\frac{\alpha_{n+m}^-}{p^+\ell} + \frac{1}{(p^+\ell)^2}\left(\frac{D-2}{12}\,n(n^2-1) + 2na\right)\delta_{n+m} \tag{15.6}$$

Exercise 16

Find values of the mass squared M^2 for the following states using the value $a = 1$:

$$\begin{aligned}
&\underline{\text{open string:}} \\
&\qquad |\psi_1\rangle = \alpha^i_{-1}|0\rangle \\
&\qquad |\psi_2\rangle = \alpha^i_{-1}\alpha^i_{-1}\alpha^j_{-2}|0\rangle \\
&\underline{\text{closed string:}} \\
&\qquad |\chi_1\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \\
&\qquad |\chi_2\rangle = \alpha^i_{-1}\alpha^i_{-1}\tilde{\alpha}^j_{-2}|0\rangle
\end{aligned} \tag{16.1}$$

Exercise 17

Consider the transverse components of the open string operator $X^{tr}(\tau, \sigma) = X^i(\tau, \sigma)$, $(i = 1, \ldots, D-2)$. We wish to compute the propagator associated with the transverse modes of an open string. Let us extend for convenience the definition of normal ordering to the operators p^i, x^i in the following way:

$$\begin{aligned}
:\alpha^i_n \alpha^j_{-n}: &= :\alpha^j_{-n}\alpha^i_n: = \alpha^j_{-n}\alpha^i_n \ ; \quad (n > 0) \\
:p^i x^j: &= :x^j p^i: = x^j p^i
\end{aligned} \tag{17.1}$$

Show that:

$$\begin{aligned}
X^i(\tau,\sigma)X^j(\tau',\sigma') &= :X^i(\tau,\sigma)X^j(\tau',\sigma'): + \delta^{ij}\, G(\tau,\, \tau',\, \sigma,\, \sigma') \\
G(\tau,\, \tau',\, \sigma,\, \sigma') &= -\frac{1}{4}\log\left[(e^{i\sigma'^{+}} - e^{i\sigma^{+}})(e^{i\sigma'^{-}} - e^{i\sigma^{-}})\right] + \\
&\quad \mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')
\end{aligned} \tag{17.2}$$

where "$\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$" denotes terms which are not divergent in the limit $\tau \to \tau', \sigma \to \sigma'$. Find the expressions of G and therefore of $\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$.
[**Hint:** *Write the open string solution $X^i(\tau, \sigma)$ in the form (taking the scale $\ell = 1$):*

$$\begin{aligned}
X^i(\tau,\sigma) &= x^i(\tau) + A^i(\tau,\sigma) + A^{i\dagger}(\tau,\sigma) \\
x^i(\tau) &= x^i + p^i\tau \\
A^i(\tau,\sigma) &= i\sum_{n=1}^{\infty}\frac{\alpha^i_n}{n}\, e^{-in\tau}\, \cos(n\sigma) \\
A^{i\dagger}(\tau,\sigma) &= -i\sum_{n=1}^{\infty}\frac{\alpha^i_{-n}}{n}\, e^{in\tau}\, \cos(n\sigma)
\end{aligned} \tag{17.3}$$

show that the only non normal ordered terms in the product $X^i(\tau,\sigma)X^j(\tau',\sigma')$ are $p^i x^j \tau$ and $A^i(\tau,\sigma)\, A^{i\dagger}(\tau',\sigma')$. These terms will be rewritten in terms of their normal *products and of the following commutators:* $[p^i, x^j]\,\tau$ *and* $\left[A^i(\tau,\sigma),\, A^{j\dagger}(\tau',\sigma')\right]$*. Using the formula:*

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \;=\; -\log\,(1-x) \tag{17.4}$$

from the expression of these commutators deduce $G(\tau,\, \tau',\, \sigma,\, \sigma')$*.]*
Show that for $\tau \neq \tau'$ and $\sigma \neq \sigma'$, $G(\tau,\, \tau',\, \sigma,\, \sigma')$ fulfills the equation of motion:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, G(\tau,\, \tau',\, \sigma,\, \sigma') \;=\; 0 \tag{17.5}$$

where the partial derivation is ment *only* with respect to τ, σ and not with respect to τ', σ'. From equations (17.2) show that the function G is the *propagator* of each transverse mode from the point (τ', σ') to the point (τ, σ), namely show that for $\tau > \tau'$:

$$\langle 0|X^i(\tau,\sigma)X^j(\tau',\sigma')|0\rangle \;=\; \delta^{ij}\, G(\tau,\, \sigma,\, \tau',\, \sigma')\;\;(\tau > \tau') \tag{17.6}$$

In the following we shall always consider $\tau > \tau'$.
We wish to show that $G(\tau,\, \tau',\, \sigma,\, \sigma')$ in the whole τ, σ plane fulfills the following equation:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, G(\tau,\, \tau',\, \sigma,\, \sigma') \;=\; i\pi\, \delta(\tau-\tau')\delta(\sigma-\sigma') \tag{17.7}$$

Only the terms in G which diverge for $\tau \to \tau'$ and $\sigma \to \sigma'$ (see equation (17.2)) contribute to the delta functions on the right hand side while the regular terms in $\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$ are fixed by other boundary conditions which we shall not consider here. Let us introduce the Green function $\tilde{G}(\tau-\tau',\, \sigma-\sigma')$ of the open string field equation:

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\, \tilde{G}(\tau-\tau',\, \sigma-\sigma') \;=\; \delta(\tau-\tau')\delta(\sigma-\sigma') \tag{17.8}$$

It is useful to express a solution of the above equation in terms of Fourier transforms with respect to the variables $\tau-\tau' \in (-\infty,\, \infty)$ and $\sigma-\sigma' \in (-\pi,\, \pi)$. Let us define the following new variables: $\xi^0 = (\tau-\tau')\Delta$, $\xi^1 = (\sigma-\sigma')\Delta$ where Δ is a scale which we shall send to infinity in order to work with coordinates ξ^α which run from $-\infty$ to $+\infty$ (indeed ξ^1 will take values in the interval $(-\pi\Delta,\, \pi\Delta)$ which becomes $(-\infty,\, \infty)$ in the limit $\Delta \to \infty$). The momenta associated with ξ^α are denoted by k^α. In particular k^1 is quantized as n/Δ (n integer) and in the limit $\Delta \to \infty$ becomes a continuous variable. Therefore the following approximation holds:

$$\sum_{n=0}^{\infty} f(\sigma-\sigma')\, e^{in(\sigma-\sigma')} \;\sim\; \Delta \int_0^\infty dk^1\, f(\xi^1/\Delta)\, e^{ik^1\xi^1} \tag{17.9}$$

Show that the following function:

$$\tilde{G}(\tau - \tau', \sigma - \sigma') = -\int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon}$$
$$\xi^\alpha = \Delta(\sigma^\alpha - \sigma'^\alpha) \qquad (17.10)$$

fulfills equ. (17.8) by using the following useful relations:

$$\int_{-\infty}^{\infty} dx e^{ixy} = 2\pi\,\delta(y)$$
$$\delta(ay) = \frac{1}{a}\delta(y) \qquad (17.11)$$

(ϵ is an infinitesimal parameter used for regularizing the integral).
Let us show that the singular parts of G and of $\tilde{G}$ as given by equ. (17.10) are proportional. In this way we would show that G is proportional to the Green function of the open string equation of motion.
We start by computing $\tilde{G}$ from equ. (17.10) as a function of the world sheet coordinates. Show that:

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon} = -\frac{i}{2(|k^1| - i\epsilon)} e^{-i|k^1|\xi^0 + ik^1\xi^1} \qquad (17.12)$$

[**Hint:** *Compute the integral in the complex k^0 plane and write the denominator as the product of two simple poles:*

$$(k^0)^2 - (k^1)^2 + i\epsilon = \left[k^0 - (|k^1| - i\epsilon)\right]\left[k^0 - (-|k^1| + i\epsilon)\right] \qquad (17.13)$$

since $\xi^0 > 0$ we should close the contour of integration in the lower plane so to include the pole $|k^1| - i\epsilon$ and then compute the integral as the residue in that pole using the formula:

$$\oint_{C_{z_0}} dz \frac{f(z)}{z - z_0} = -2\pi i f(z_0) \qquad (17.14)$$

where $f(z)$ is a regular function in z_0 and C_{z_0} is a contour around z_0 oriented clockwise.]
We are left with the integral over k^1 which is divergent in $k^1 = 0$ in the limit $\epsilon \to 0$. It is useful to express this integral in terms of its principal part by using the relation:

$$\frac{1}{z \pm i\epsilon} = \text{PP}\left(\frac{1}{z}\right) \mp i\delta(z) \qquad (17.15)$$

Show that the second term on the right hand side contributes to $\tilde{G}$ with a constant term. Since we are interested only in the singular part of $\tilde{G}$ we shall ignore this term and simply substitute the integral in k^1 with its principal part which can be expressed as follows:

$$\mathrm{PP}\int_{-\infty}^{\infty} dk^1\, f(k^1) \;=\; \lim_{\Delta\to\infty}\left(\int_{1/\Delta}^{\infty} dk^1\, f(k^1) + \int_{-\infty}^{-1/\Delta} dk^1\, f(k^1)\right)$$

Show that:

$$\begin{aligned}\tilde{G}(\tau-\tau',\,\sigma-\sigma') &= \mathrm{PP}\int_{-\infty}^{\infty}\frac{dk^1}{2\pi}\,\frac{i}{2|k^1|}\,e^{-i|k^1|\xi^0+ik^1\xi^1}\\ &= \frac{i}{4\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-in(\sigma^- - \sigma'^-)} + \sum_{n=1}^{\infty}\frac{1}{n}\,e^{-in(\sigma^+ - \sigma'^+)}\right) \qquad (17.16)\end{aligned}$$

[**Hint:** *Use the property that* $k^1 = n/\Delta$ *and therefore one has*

$$\int_{1/\Delta}^{\infty} dk^1 \;\sim\; (1/\Delta)\sum_{n=1}^{\infty}$$

for large Δ.]

From (17.16) show that the functions $\tilde{G}$ and G are proportional a part from terms which are not divergent in the limit $\tau \to \tau'$ and $\sigma \to \sigma'$ and from this deduce that equation (17.7) holds.

Exercise 18

Show that:

$$\begin{aligned} :e^{ik_1^i X^i(\tau,0)}::e^{ik_2^i X^i(0,0)}: \;&=\; :e^{ik_1^i X^i(\tau,0)}\,e^{ik_2^i X^i(\tau,0)}:\; e^{-(k_1\cdot k_2)\,G(\tau,0,0,0)} =\\ &\quad (1-x)^{k_1\cdot k_2}\,x^{-k_1\cdot k_2}\\ x \;&=\; e^{-i\tau} \qquad (18.1)\end{aligned}$$

by using the expression for the propagator G found in the previous exercise and the basic formulas in section 6 of the lecture notes.

Exercise 19

Anticommuting c-numbers. Consider anticommuting c-numbers θ_i, satisfying

$$\theta_i\theta_j = -\theta_j\theta_i\,.$$

The numbers θ_i are taken to be real, i.e. $\theta_i^\dagger = \theta_i$.

- Define a function $F(\theta_1, \theta_2, \theta_3)$, where the θ_i are 3 anticommuting variables, called *Grassmann* variables. Show that there are 8 terms (*monomials*) in the polynomial decomposition of the function in terms of the θ variables. How many independent terms do you expect in the decomposition of a function of n Grassmann variables? We first look at differentiation. The *left* derivative of a function is obtained by differentiating its monomials and resumming the result. To calculate the left derivative with respect to θ_i we must, in every monomial, permute θ_i to the left and then drop it. Let $\epsilon[i]$ be the sign of the permutation needed to bring θ_i to the left, and $\epsilon[i] = 0$ when θ_i does not occur in the monomial. The left derivative of a monomial can then be written in the following way,

$$\frac{\overrightarrow{\partial}}{\partial\theta_{i_k}}\left(\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_k}\theta_{i_{k+1}}\cdots\theta_{i_m}\right) = \epsilon[i_k]\left(\theta_{i_1}\cdots\theta_{i_{k-1}}\theta_{i_{k+1}}\cdots\theta_{i_m}\right). \tag{19.1}$$

  Analogously, one can define a *right* derivative.

- Define a function $F(\theta_1, \theta_2)$, calculate

$$\frac{\overrightarrow{\partial}}{\partial\theta_1}\;F(\theta_1,\theta_2),\;\text{and}$$
$$\frac{\overrightarrow{\partial}}{\partial\theta_2}\;F(\theta_1,\theta_2).$$

  The definition of the left derivative corresponds to the variational statement

$$\delta_\theta F(\theta) = \delta\theta\,\frac{\overrightarrow{\partial}}{\partial\theta}\,F(\theta), \tag{19.2}$$

  where $\delta\theta$ is a Grassmann parameter. Note the order of the terms in the above expression.

- For the right derivative write down the corresponding variational statement.
- Consider a $\psi(\theta_i)$ which is a function of the θ_i $(i = 1, \ldots, n)$ variables. Work out

$$\frac{\overrightarrow{\partial}}{\partial\theta_j}\,F(\psi(\theta_i));\quad \frac{\overleftarrow{\partial}}{\partial\theta_j}\,(F(\psi(\theta_i)))\,.$$

Again, the order in which the terms appear is important. This is an immediate consequence of working with anticommuting variables. Now look at Leibniz' rule:

- Work out

$$\frac{\vec{\partial}}{\partial \theta}\left(F(\theta)\ G(\theta)\right).$$

- Calculate

$$\frac{\vec{\partial}}{\partial \theta}\exp(\theta)$$

We now construct the analogue of the indefinite one-dimensional integral,

$$\int_{-\infty}^{\infty} dx\ f(x),$$

for the case of anticommuting c-numbers, which we denote by

$$\int d\theta F(\theta).$$

We want it to obey the following property of the integral over commuting variables,

$$\int_{-\infty}^{\infty} dx\ f(x) = \int_{-\infty}^{\infty} dx\ f(x+a) \tag{19.3}$$

with a finite.

- Consider a function $F(\theta)$ of one Grassmann variable θ. Show that requiring the property (19.3) to hold leads to the requirement

$$\int d\theta\ [\text{any element not depending on } \theta] = 0.$$

Compare with differentiating a function of commuting variables. We are then left with an integral over θ, which we normalize to unity,

$$\int d\theta\ \theta \equiv 1.$$

Exercise 20

Let us consider the change of variables from τ, σ to z, $\overline{z}$. After a Wick rotation we write $\tau = i\sigma_2$ where σ_2 is the real Euclidean time and we rename σ as σ_1. If we define $w = \sigma^+ = i\sigma_2 + \sigma_1$ so that $\overline{w} = -\sigma^- = -i\sigma_2 + \sigma_1$ then z, $\overline{z}$ are defined as follows:

$$\begin{array}{lll} \underline{\text{open string:}} & z = e^{iw} \;; & \overline{z} = e^{-i\overline{w}} \\ \underline{\text{closed string:}} & z = e^{2iw} \;; & \overline{z} = e^{-2i\overline{w}} \end{array} \tag{20.1}$$

- For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by z for the open and closed string cases and draw the equal–time curves. Which points correspond to $\sigma_2 \to \pm\infty$?
- Write as functions of the complex variables the open and closed string solutions $X^\mu(z, \overline{z})$ together with their holomorphic and anti–holomorphic derivatives $\partial_z X^\mu(z, \overline{z})$, $\partial_{\overline{z}} X^\mu(z, \overline{z})$.
- Rewrite the Green function $G(\tau, \tau', \sigma, \sigma')$ computed in exercise 17 as a function $G(z, \overline{z}, z', \overline{z}')$ in the complex variables. Compute it for the closed string as well.
- Consider the variable $\hat{z} = \log(z)$. For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by $\hat{z}$ for the open and closed string cases and draw the equal–time curves.

Exercise 21

Let us consider conformal transformations in the complex coordinate notation. In these coordinates conformal transformations have the form: $z \to z'(z)$ and $\overline{z} \to \overline{z}'(\overline{z})$. We shall denote in the sequel by $\Phi_{h,\bar{h}}(z, \overline{z})$ a tensor with h holomorphic and $\bar{h}$ anti–holomorphic lower indices. Under a conformal transformation $\Phi_{h,\bar{h}}$ transforms as follows:

$$\Phi_{h,\bar{h}}(z, \overline{z}) \quad \to \quad \Phi'_{h,\bar{h}}(z, \overline{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{\bar{h}} \Phi_{h,\bar{h}}(z'(z), \overline{z}'(\overline{z})) \tag{21.1}$$

- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to e^{i\theta} z$, $\overline{z} \to e^{-i\theta}\, \overline{z}$ where θ is a constant angle.
- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to z' = \log(z)$, $\overline{z} \to \overline{z}' = \log(\overline{z})$.

Consider an infinitesimal conformal transformation $z \to z' = z + \epsilon(z)$ and $\overline{z} \to \overline{z}' = \overline{z} + \overline{\epsilon}(\overline{z})$. Show that:

$$\begin{aligned} \Phi'_{h,\bar{h}} &= \Phi_{h,\bar{h}} + \delta\Phi_{h,\bar{h}} \\ \delta\Phi_{h,\bar{h}} &= \left(h\,(\partial\epsilon) + \bar{h}\,(\overline{\partial}\overline{\epsilon}) + \epsilon\,\partial + \overline{\epsilon}\,\overline{\partial}\right) \Phi_{h,\bar{h}} \end{aligned} \tag{21.2}$$

where we have used the notation: $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial \overline{z}$.

Consider now the quantized closed string and let $\Phi_{h,\bar{h}}$ be an operator on the Hilbert space of states which transforms under conformal transformations as in eq (21.1) (*primary operator*) . We wish to define generators of infinitesimal conformal transformations T_ϵ, $\overline{T}_{\overline{\epsilon}}$ on the Hilbert space such that:

$$\delta\Phi_{h,\bar{h}} \;=\; [T_\epsilon,\, \Phi_{h,\bar{h}}] + [\overline{T}_{\overline{\epsilon}},\, \Phi_{h,\bar{h}}] \tag{21.3}$$

Let us consider the Laurent expansion of ϵ, $\overline{\epsilon}$:

$$\epsilon = \sum_{n=-\infty}^{\infty} \epsilon_n\, z^{n+1} \qquad \overline{\epsilon} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \overline{z}^{n+1}$$

and define T_ϵ, $\overline{T}_{\overline{\epsilon}}$ in terms of some operators ℓ_n and $\bar{\ell}_n$ as follows:

$$T_\epsilon \;=\; \sum_{n=-\infty}^{\infty} \epsilon_n\, \ell_n \quad \overline{T}_{\overline{\epsilon}} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \bar{\ell}_n \tag{21.4}$$

Show that if the following commutation relations hold:

$$\begin{aligned} \left[\ell_n,\, \Phi_{h,\bar{h}}\right] &= z^n(z\partial + h(n+1))\Phi_{h,\bar{h}} \\ \left[\bar{\ell}_n,\, \Phi_{h,\bar{h}}\right] &= \overline{z}^n(\overline{z}\partial + \bar{h}(n+1))\Phi_{h,\bar{h}} \end{aligned} \tag{21.5}$$

formula (21.3) yields the transformation rule (21.2). Compare formulas (21.5) and (21.4) with the analogous formulae of exercise 14. Observe the differences with respect to exercise 14: there conformal transformations were considered for simplicity only on scalar functions Y (which have $h = \bar{h} = 0$); the infinitesimal generator had the same form as in (21.4), but its action on the function Y is a differential operation, while in the present exercise its action on operators is expressed in terms of commutators (21.5); the light cone indices $\pm$ are substituted by holomorphic/anti–holomorphic indices in the present exercise.

Operators on free closed string states are expressed in terms of α_n^i and $\tilde{\alpha}_n^i$. An asymptotic string state emitted from a point $z, \overline{z}$ of the world sheet is described by a local *primary* operator $V(z, \overline{z})$ (i.e. transforming as in equ.(21.1) with definite values of h, $\bar{h}$) which is called *vertex operator*.

Consider the vertex operator $V(z,\overline{z})$ =: $e^{ik^i X^i(z,\overline{z})}$: (from now on, for the vertex operators, we shall consider only space–like momenta $k^\mu \equiv k^i$) describing the emission of a tachyon of momentum k^i (verify indeed that $[p^i, V] = k^i\, V$). If for ℓ_n and $\bar{\ell}_n$ we take $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ defined in exercise 15, show that the commutator of these operators with V have the expression on the right hand side of eqs. (21.5) for suitable values of h, $\bar{h}$. Find these values. Repeat this exercise for the vertex operator $V(z, \overline{z})$ =: $\partial X^j\, \overline{\partial} X^i\, e^{ik\cdot X(z,\overline{z})}$:.

In general infinitesimal conformal transformations on the closed string Hilbert space are generated by T_ϵ, $\overline{T}_{\overline{\epsilon}}$ which are expressed in terms of $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$.

There is a remark to be done about conformal transformations and the light–cone gauge. In the light–cone gauge the reparametrization invariance of the string action is totally fixed and with it also the conformal transformations which are generated by T_{++} and T_{--}. Indeed on the Hilbert space the Fourier modes of T_{++} and T_{--}, which are L_n and $\tilde{L}_n$ and are the generators of the conformal transformations are fixed to the value $\delta_{n,0}$ (see exercise 15). The generators $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ have the same expression as L_n and $\tilde{L}_n$ but involve only oscillators along transverse directions. They close a Virasoro algebra as well but with a different central charge given by $c = D - 2$, and thus are generators of conformal transformations though we do not expect the action to be invariant with respect to it (since all reparametrization invariance have been fixed).

In the framework of covariant quantization the constraint $T_{\alpha\beta} = 0$ is imposed on the physical states and translated into the conditions: $L_n|phys\rangle = \tilde{L}_n|phys\rangle = 0$ $(n > 0)$; $(L_0 - 1)|phys\rangle = (\tilde{L}_0 - 1)|phys\rangle = 0$. To summarize the $\{L_n,\, \tilde{L}_n\}$ and $\{\mathcal{L}_n,\, \tilde{\mathcal{L}}_n\}$ conformal algebras are different and with respect to them physical states will have different conformal weights $\{h,\, \bar{h}\}$. In particular the conformal weights with respect to the first are bound by the constraints written above to be $(1, 1)$, while the conformal weights corresponding to the action of the second algebra are not constrained.

Exercise 22

There is a one to one correspondence between vertex operators $V(z, \overline{z})$ and asymptotic in–coming or our–going states of a string (free states). The in–coming string state $|V,\, in\rangle$ associated with the operator $V(z, \overline{z})$ is defined through the asymptotic limit $\sigma_2 \to \infty$ of $V(z, \overline{z})|0\rangle$ (recall that $|0\rangle$ is the vacuum state with vanishing CM momentum), similarly the corresponding out–going free state $\langle V,\, out|$ is obtained by performing on the state $\langle 0|V(z, \overline{z})$ the opposite limit $\sigma_2 \to -\infty$. Use the complex notation and find the in–coming and out–going states corresponding to the following vertex operators (for simplicity in what follows the momentum k^μ is always space-like):

$$\underline{\text{open string:}} \qquad : e^{ik^i X^i(z,\overline{z})} : \qquad : \partial\, X^j\, e^{ik\cdot X(z,\overline{z})} :$$

$$\underline{\text{closed string:}} \qquad : e^{ik^i X^i(z,\overline{z})} : \qquad : \partial\, X^j\, \overline{\partial}\, X^i\, e^{ik\cdot X(z,\overline{z})} : \tag{22.1}$$

Compute the mass squared M^2 of the in–coming asymptotic states computed above either through direct evaluation of M^2 on $|V,\, in\rangle$ or by performing the limit $\sigma_2 \to \infty$ on $M^2(V(z, \overline{z})|0\rangle)$.

[**Hint:** *Express* $M^2(V(z, \overline{z})|0\rangle)$ *as* $[M^2,\, V(z, \overline{z})]\,|0\rangle + V(z, \overline{z})\, M^2|0\rangle$]

Exercise 23

Representations of the Clifford algebra. The Clifford algebra of Dirac matrices:

$$\{\gamma^\mu \gamma^\nu\} = -2\eta^{\mu\nu} \tag{23.1}$$

cannot be represented in any dimensions. Show, that for representing the γ^μ matrices in $2n$ spacetime dimensions we need at least $2^n \times 2^n$ matrices. To do that, construct n linear operators out of the γ matrices, eg.

$$a_i := \frac{1}{\sqrt{2}}(\gamma_{2i-2} + i\gamma_{2i-1}) \quad i = 1..n$$

use a representation[1] for which

$$\gamma^\dagger = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i \quad i = 1..2n-1.$$

Show that

$$\{a_n, a_m^\dagger\} = \delta_{m,n},$$

define a vacuum $|0\rangle$ and determine the number of different states which the creation operators can create from the vacuum. If we want to impose further conditions (like Weyl or Majorana spinor) we will find further constraints. Check that for two dimensions the following definition

$$\gamma^0 = \sigma^2 \quad \gamma^1 = i\sigma^1$$

(where σ^i means the i-th Pauli matrix) satisfies the algebra (23.1). Now with an inductive procedure we can construct a representation of 2^n dimensions if the spacetime has $2n$ dimensions (that is we will have 2n $2^n \times 2^n$ matrices). Suppose that we have the algebra for $d = 2n-2$.($\hat{\gamma}^\mu$'s are given for $\mu = 0...2n-3$ and (23.1) is satisfied). Then we define two more matrices according to the following:

$$\gamma^{2n-2} = i\mathbf{1} \times \sigma^1 \quad \gamma^{2n-1} = i\mathbf{1} \times \sigma^2 \tag{23.2}$$

and extend the first $2n-2$ as

$$\gamma^\mu = \hat{\gamma}^\mu \times (-\sigma_3) \quad \mu = 1..2n-1$$

$\mathbf{1}$ is the unit matrix in 2^{n-1} dimensions. ($(A \times B)_{ijkl} = A_{ik}B_{jl}$ is the formula for the indices of the direct product matrix.) That is these two matrices are blockdiagonal ones, each entry of the appropriate Pauli matrix is the entry times the $2^{n-1} \times 2^{n-1}$ $\hat{\gamma}^\mu$ or unit matrix. Check that (23.1) is then satisfied in 2n dimensions. What about the case of odd dimensions ? Show that in $2n+1$ dimensions $2^n \times 2^n$ matrices still can represent the algebra. [**Hint:** *Extend the case of* $2n$ *with* $\gamma^{2n} = \gamma$ *defined as (23.5) further in the exercise.*]

[1] One needs to prove that this representation always exists.

Since the right hand side of (23.1) is real the relation for the anticommutator of the complex conjugate γ's are the same:

$$\{\gamma^{\mu *}\gamma^{\nu *}\} = -2\eta^{\mu\nu}$$

The complex conjugate representation is equivalent, that is

$$\gamma^{\mu *} = B\gamma^{\mu}B^{-1} \quad \mu = 1...2n-1 \tag{23.3}$$

where both B_1 and B_2 are unitary. An explicit construction for such a matrix is the following:

$$B_1 = \gamma^3\gamma^5...\gamma^{2n-1} \quad B_2 = \gamma^0\gamma^1\gamma^2\gamma^4...\gamma^{2n-2} \tag{23.4}$$

Show that

$$B_1\gamma^{\mu}B_1^{-1} = (-1)^n\gamma^{\mu *} \quad B_2\gamma^{\mu}B_2^{-1} = (-1)^{n-1}\gamma^{\mu *}$$

that is one of them always satisfies (23.3).

Define now

$$\gamma = c\gamma^0\gamma^1...\gamma^{2n-1} \tag{23.5}$$

Determine the value of c by the requirement that $(\gamma)^2 = \mathbf{1}$. Can we generalize the relations (23.4) to odd $(2n+1)$ dimensions ?

The spinors on which the γ^{μ} matrices act are representations of the Lorentz group with the generator

$$\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$$

where the bracket is the commutator. With respect to the full Lorentz group these are not always irreducible representations, the familiar projections,

$$P_L = \frac{1-\gamma}{2} \quad P_R = \frac{1+\gamma}{2}$$

project out irreducible subspaces. In physical language we can find chiral (left and right handed) fermions, which are not mixed under the action of the Lorentz group. They are then called *Weyl spinors.*

When can one find real spinor representations ? In other words one wants to impose the following condition

$$\psi = B\psi^* \tag{23.6}$$

Show that ψ and $B\psi^*$ transforms according to the same representation of the Lorentz group. Show, that from the above condition requirement $B^*B = 1$ follows. Verify the following formulae for the explicit B_1 and B_2:

$$B_1B_1^* = (-1)^{\frac{n(n-1)}{2}} \quad B_2B_2^* = (-1)^{\frac{(n-1)(n-2)}{2}}$$

The spinors for which (23.6) holds are called *Majorana spinors.*

Verify that

$$B_i(\frac{1 \pm \gamma}{2}\psi)^* = \frac{1 \pm (-1)^{(n+1)}}{2} B_i \psi^* \quad i = 1, 2$$

that is the in certain spacetime dimensions

$$(\psi_L)^* = \psi_L \quad (\psi_R)^* = \psi_R$$

that is in certain spacetime dimensions $\psi_L^* = \psi_L$, $\psi_R^* = \psi_R^*$. In which spacetime dimensions are there Weyl, Majorana, Weyl-Majorana spinors ? [**Hint:** *Use the given explicit representation of B's and* γ^μ*'s.*]

Exercise 24

Check explicitly the calculations leading to (11.22) and (11.23) of the lecture notes.

Exercise 25

Supersymmetric particle. Consider the action of a massless *super particle* propagating in D–dimensional Minkowski space, which is obtained from the particle world line action by adding to the D bosonic fields $x^\mu(\tau)$, the contribution from D Majorana fermions $\psi^\mu(\tau)$ ($\psi^{\mu\star} = \psi^\mu$) which describe the corresponding *super partner*:

$$S = -\frac{1}{2} \int d\tau \, e \left((\dot{x}^\mu)^2 - i\psi^\mu \dot{\psi}_\mu \right) \tag{25.1}$$

Compute the field equations for the fields x^μ, ψ^μ, e and show that the field equation for e amounts to a constraint on the other fields. Consider the following super symmetry transformation:

$$\delta x^\mu \;=\; i\epsilon\psi^\mu \;\; ; \;\; \delta\psi^\mu = \epsilon\dot{x}^\mu \tag{25.2}$$

where ϵ is an infinitesimal real Grassmann parameter (recall that on Grassmann variables the complex conjugation is defined to have the property that $(\epsilon_1\epsilon_2)^* = \epsilon_2^*\epsilon_1^*$ just like Hermitian conjugation for matrices. Therefore you can show that the product of two real Grassmann variables acts as an imaginary c–number). The fields x^μ and ψ^μ are called super partners because they transform into each other by super symmetry. If we denote by δ_1 and δ_2 two infinitesimal super symmetry transformations parametrized by ϵ_1 and ϵ_2 respectively, using (25.2), show that:

$$\delta_1(\delta_2 x^\mu) - \delta_2(\delta_1 x^\mu) \;=\; \delta\tau \, \frac{dx^\mu}{d\tau} \tag{25.3}$$

and find the expression of $\delta\tau$. From the above result we see that the commutator of two super symmetry transformations amounts to a space–time coordinate translation. If the action were invariant under *local* super symmetry then (25.3) would guarantee its invariance under local reparametrization as well. Show that under the transformation (25.2) if ϵ depends on τ we can write:

$$\delta S \;=\; -2\int d\tau\,\left(i\dot{\epsilon}J - \frac{ie}{2}\frac{d}{d\tau}(\epsilon\dot{x}\psi) + \text{field equations}\right) \tag{25.4}$$

and find the expression for J. Is the action S invariant under global super symmetry transformations (i.e. $\dot{\epsilon} = 0$)? Under which constraint is the action invariant under local super symmetry transformations? Since our action is invariant under local reparametrization, in view of (25.3), we would like this invariance to be a consequence of local super symmetry, then we shall impose the local super symmetry constraint computed above. This constraint, as explained in the lecture notes, allows to have finite fermionic mass spectrum, and the consequent local super symmetry will allow to gauge away non–physical longitudinal modes of ψ^μ.

Consider the canonical quantization of the super particle setting for simplicity $e = 1$. Show form the canonical commutation relations that $\{\psi^\mu(\tau),\, \psi^\nu(\tau)\} = \eta^{\mu\nu}$.

[**Hint:** *In order to work with a well defined canonical momentum* Π_ψ *associated with* ψ *it is advisable to follow the procedure described in section 11.4 of the lecture notes.*]
What can you conclude about the states generated by ψ^μ?

We can construct a locally super symmetric action by introducing a new auxiliary Grassmann valued field ν to be regarded as the super partner of the auxiliary field e and modifying the action as follows:

$$S \;=\; -\frac{1}{2}\int d\tau\,\left(e(\dot{x}^\mu)^2 - ie\psi^\mu\dot{\psi}_\mu - 2i\nu\dot{x}^\mu\psi_\mu\right) \tag{25.5}$$

Show that this action is invariant under the following local super symmetry transformations:

$$\delta x^\mu = i\epsilon\psi^\mu\,;\quad \delta\psi^\mu = \epsilon\dot{x}^\mu\,;\quad \delta e = -2i\epsilon\nu\,;\quad \delta\nu = \dot{\epsilon}e - \frac{1}{2}\epsilon\dot{e}\,. \tag{25.6}$$

Compute the field equations for the various fields and verify that the field equation for ν yields the local super symmetry constraint derived previously. If we regard the physical states of this theory as space–time fields what condition does the local super symmetry constraint imply on these fields?

Locally super symmetric theories are called super gravity theories since they include invariance under general coordinate transformation, which is the symmetry of General Relativity, as a consequence of local super symmetry. The field e is called the *graviton* and its superpartner ν the *gravitino*. In one and two dimensions both these fields are non–propagating.

Exercise 26

Consider the superstring action (eq. (11.45) of the lecture notes):

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{26.1}$$

Show that under the super symmetry trnsformations in eq. (11.44) of the lecture notes the action transforms as in (11.46). What is the constraint for *local* super symmetry?

Also in this case this constraint can be derived from a locally super symmetric action as the field equation of an auxiliary gravitino χ^α, super partner of the graviton e_α^a. The construction of this action follows the same lines as in the super symmetric particle case, i.e. with the addition of a term of the form $\overline{\chi}^\alpha J_\alpha$. In this case however the addition of a further quadratic term in the gravitino field is required by local super symmetry invariance. The derivation of this action is described in detail in section 4.3.5 of the Green–Schwarz–Witten book. It is lengthy and we shall not deal with it in this exercise. However there are some properties of Majorana spinors in two dimensions which are needed for this derivation and are useful to derive. Show that:

$$\overline{\psi}^\mu \rho^\alpha \nabla_\alpha \psi_\mu = \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\,; \qquad \psi_A^\mu \overline{\psi}_{B\mu} = -\frac{1}{2}\overline{\psi}_\mu \psi^\mu\, \delta_{AB}\,. \tag{26.2}$$

Exercise 27

Given local left–moving and right–moving fields $\Phi(\tau+\sigma)$, $\tilde{\Phi}(\tau-\sigma)$ on the world sheet of a string ($\Phi \equiv \tilde{\Phi}$ for the open string), which are expanded in Fourier modes as follows:

$$\begin{aligned}
\underline{\text{open string:}} \qquad \Phi(\tau\pm\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-in(\tau\pm\sigma)} \\
\underline{\text{closed string:}} \qquad \Phi(\tau+\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-2in(\tau+\sigma)} \\
\tilde{\Phi}(\tau-\sigma) &= \frac{1}{\Delta}\sum_n \tilde{\Phi}_n\, e^{-2in(\tau-\sigma)}
\end{aligned} \tag{27.1}$$

where Δ is a normalization factor. Show that the following inverse relations hold:

$$\begin{aligned}
\underline{\text{open string:}} \qquad \Phi_n &= \frac{\Delta}{2\pi}\int_0^\pi d\sigma\, \left[\Phi(\tau-\sigma)\, e^{in(\tau-\sigma)} + \Phi(\tau+\sigma)\, e^{in(\tau+\sigma)}\right] \\
\underline{\text{closed string:}} \qquad \Phi_n &= \frac{\Delta}{\pi}\int_0^\pi d\sigma\, \Phi(\tau+\sigma)\, e^{2in(\tau+\sigma)} \\
\tilde{\Phi}_n &= \frac{\Delta}{\pi}\int_0^\pi d\sigma\, \tilde{\Phi}(\tau-\sigma)\, e^{2in(\tau-\sigma)}
\end{aligned} \tag{27.2}$$

Let Φ and $\tilde{\Phi}$ be fermionic fields with Neveu-Schwarz boundary conditions, express them as Fourier series and find the inverse transformations yielding the coefficients. Using the above relations derive the expressions for the Fourier coefficients α_n^μ, d_n^μ, b_r^μ, L_n, F_n, G_r as functions of $\partial_+ X^\mu$, ψ_+^μ(Ramond), ψ_+^μ (Neveu–Schwarz), T_{++}, J_+ (Ramond), J_+ (Neveu–Schwarz) respectively and the corresponding left–handed quantities, for open and closed strings. Recall that:

<u>open string:</u>

$$\begin{aligned}
\partial_\pm X^\mu &= \frac{1}{2}\sum_n \alpha_n^\mu e^{-in(\tau\pm\sigma)} \\
\psi_\pm^\mu\,(\mathrm{R}) &= \frac{1}{\sqrt{2}}\sum_n d_n^\mu e^{-in(\tau\pm\sigma)} \\
\psi_\pm^\mu\,(\mathrm{NS}) &= \frac{1}{\sqrt{2}}\sum_{r\in Z+1/2} b_r^\mu e^{-ir(\tau\pm\sigma)} \\
T_{\pm\pm} &= \frac{1}{2}\sum_n L_n\, e^{-in(\tau\pm\sigma)} \\
J_\pm\,(\mathrm{R}) &= \frac{1}{2\sqrt{2}}\sum_n F_n\, e^{-in(\tau\pm\sigma)} \\
J_\pm\,(\mathrm{NS}) &= \frac{1}{2\sqrt{2}}\sum_{r\in Z+1/2} G_r\, e^{-ir(\tau\pm\sigma)}
\end{aligned}$$

<u>closed string:</u>

$$\begin{aligned}
\partial_+ X^\mu &= \sum_n \alpha_n^\mu e^{-2in(\tau+\sigma)} & \partial_- X^\mu &= \sum_n \tilde{\alpha}_n^\mu e^{-2in(\tau-\sigma)} \\
\psi_+^\mu\,(\mathrm{R}) &= \sum_n d_n^\mu e^{-2in(\tau+\sigma)} & \psi_-^\mu\,(\mathrm{R}) &= \sum_n \tilde{d}_n^\mu e^{-2in(\tau-\sigma)} \\
\psi_+^\mu\,(\mathrm{NS}) &= \sum_{r\in Z+1/2} b_r^\mu e^{-2ir(\tau+\sigma)} & \psi_-^\mu\,(\mathrm{NS}) &= \sum_{r\in Z+1/2} \tilde{b}_r^\mu e^{-2ir(\tau-\sigma)} \\
T_{++} &= 2\sum_n L_n\, e^{-2in(\tau+\sigma)} & T_{--} &= 2\sum_n \tilde{L}_n\, e^{-2in(\tau-\sigma)} \\
J_+\,(\mathrm{R}) &= \sum_n F_n\, e^{-2in(\tau+\sigma)} & J_-\,(\mathrm{R}) &= \sum_n \tilde{F}_n\, e^{-2in(\tau-\sigma)} \\
J_+\,(\mathrm{NS}) &= \sum_{r\in Z+1/2} G_r\, e^{-2ir(\tau+\sigma)} & J_-\,(\mathrm{NS}) &= \sum_{r\in Z+1/2} \tilde{G}_r\, e^{-2ir(\tau-\sigma)}
\end{aligned}$$

Use the above definitions to express the constraints L_n, F_n, G_r and the corresponding tilded quantities in terms of field coefficients for both open and closed strings.

Exercise 28

From the canonical commutation/anti–commutation relations:

$$\begin{aligned}
\left[\dot{X}^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\left[X^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= \left[\dot{X}^\mu(\sigma,\tau),\, \dot{X}^\nu(\sigma',\tau)\right] = 0 \\
\{\psi^\mu_\pm(\sigma,\tau),\, \psi^\nu_\pm(\sigma',\tau)\} &= \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\{\psi^\mu_\pm(\sigma,\tau),\, \psi^\nu_\mp(\sigma',\tau)\} &= 0
\end{aligned} \tag{28.1}$$

using the results and conventions of the previous exercise, deduce the commutation/anti–commutation relations among α^μ_n, d^μ_n, b^μ_r and the corresponding left–moving quantities (see eqs. (11.39), (11.40) of the lecture notes) as operators in the *unconstrained* Hilbert space. Write the constraints L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$ as normal ordered operators (recall that the definition of normal order for fermionic operators is the same as in the bosonic case except for a minus sign required every time the order of two operators is inverted: $:d_n d_{-n}: = -d_{-n}d_n = -:d_{-n}d_n:$). Deduce the commutation/anti–commutation relations among L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$.

When a constraint is expressed as a quadratic function of quantities which, as operators, do not commute (bosonic operators) or anti–commute (fermionic operators), then their implementation at the quantum level suffers from an ordering ambiguity and therefore requires the introduction of a constant. For the bosonic string we saw that the only constraint with this ambiguity was L_0 and therefore a constant a was introduced so that in the light–cone quantization the constraints had the form $L_n - a\delta_n = 0$. In the superstring case, which of the constraints requires an ordering constant in the R and NS sectors? The ordering constants in the two sectors are in general unrelated. Consider the R sector and suppose we implement the constraint F_0 in the form $F_0 - \mu = 0$, μ being a c–number, find the relation between the constant a for L_0 and μ (use the relation $\{F_0,\, F_0\} \sim L_0$). Is it consistent to set $\mu = 0$? What consequence dose this have on the value of a for the R sector?

Let us denote by $|0\rangle_R$ and $|0\rangle_{NS}$ the vacua of the R and NS sectors (for the open string or for a single left or right mover sector of the closed string) respectively, show that $|0\rangle_{NS}$ has degeneracy one while for $D = 2n$ $|0\rangle_R$ has degeneracy 2^n and therefore describes a fermionic state (show that the action of d^μ_0 on $|0\rangle_R$ does not change the energy of the state and use the anti–commutation relations among the d^μ_0.). How many sectors does the closed string have? Write the vacua for each closed string sector.

Exercise 29

Light–cone quantization. Consider the gauge fixing conditions $X^+ = x^+ + p^+\tau$ and $\psi^+ = 0$. Show that they are consistent with supersymmetry transformations (equations (11.44) of the lecture notes). Solve them in terms of the α_n^+, d_n^+, b_r^+, $\tilde{\alpha}_n^+$, $\tilde{d}_n^+$, $\tilde{b}_r^+$ in the R, NS, left and right–moving sectors. Use these conditions to solve the constraints by expressing α_n^-, d_n^-, b_r^-, $\tilde{\alpha}_n^-$, $\tilde{d}_n^-$, $\tilde{b}_r^-$ in terms of the corresponding transverse components for the relevant sectors in the open and closed string cases using $a = 0$ in the R sector (see previous exercise) and $a = 1/2$ in the NS sector (recall that $\alpha_0^\mu = p^\mu$ for the open string and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu/2$ for the closed string). Write the formula for M^2 in the various sectors of the open and closed string. Write explicitly the tachyonic and massless open and closed string states.

Exercise 30

Find the expression of the vacuum expectation value $G^{F\,\mu\nu}$ of the product of two fermionic fields:

$$\langle 0|\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma')|0\rangle \;=\; G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma') \tag{30.1}$$

as for the bosonic case, the function G^F is the Green function of the fermionic field equation and, when time ordered, yields the fermion propagator.
[**Hint:** *Express the product of the two fields in terms of its normal ordered expression:*

$$\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma') =: \psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma') : + \,G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma')$$

]

Exercise 31

On $SO(1,9)$ and $SO(8)$. The groups $SO(1,9)$ and $SO(8)$ are defined as those groups of transformations whose action on the corresponding vector representations V^μ, v^i ($\mu = 0,\ldots,9$, $i = 1,\ldots,8$) leaves the metrics $\eta_{\mu\nu}$ and δ_{ij} invariant:

$$U^\mu{}_\nu \;\in\; SO(1,9) \;\Leftrightarrow\; U^\rho{}_\mu\,\eta_{\rho\sigma}\,U^\sigma{}_\nu \;=\; \eta_{\mu\nu} \tag{31.1}$$

$$\tilde{U}^i{}_j \;\in\; SO(8) \;\Leftrightarrow\; \tilde{U}^k{}_i\,\delta_{kn}\,\tilde{U}^n{}_j \;=\; \delta_{ij} \tag{31.2}$$

$$\eta^{\mu\nu} \;=\; \mathrm{diag}(-,\,+,\,\ldots,\,+)$$

The vector representation of $SO(8)$, described by v^i, is usually denoted by $\mathbf{8}_v$. Let us define the following matrices:

$$\mathbb{M}^\mu{}_\nu \;=\; w_{\rho\sigma}\,(M^{\rho\sigma})^\mu{}_\nu \;;\; (M^{\rho\sigma})^\mu{}_\nu \;=\; \eta^{\sigma\mu}\,\delta^\rho_\nu - \eta^{\sigma\nu}\,\delta^\rho_\mu$$

$$\tilde{\mathbb{M}}^i{}_j \;=\; w_{kn}\,(\tilde{M}^{kn})^i{}_j \;;\; (\tilde{M}^{kn})^i{}_j \;=\; \delta^{ni}\,\delta^k_j - \delta^{ki}\,\delta^n_j \tag{31.3}$$

where $w_{\mu\nu}$ and its restriction w_{ij} to the i, j indices are antisymmetric matrices of parameters. Show that $\mathbb{M}^{\mu}{}_{\nu}$ and $\tilde{\mathbb{M}}^{i}{}_{j}$ are infinitesimal generators of $SO(1,9)$ and $SO(8)$ respectively in their vector representations, i.e. that $U^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \mathbb{M}^{\mu}{}_{\nu}$ and $\tilde{U}^{i}{}_{j} = \delta^{i}{}_{j} + \tilde{\mathbb{M}}^{i}{}_{j}$ fulfill equations (31.1) and (31.2) respectively to first order in w. How many independent generators do $SO(1,9)$ and $SO(8)$ have? Using the definitions (31.3) show that the $SO(1,9)$ infinitesimal generators $M^{\mu\nu}$ fulfill the following commutation relations:

$$[M^{\mu\nu},\, M^{\rho\sigma}] \;=\; \eta^{\mu\rho}\, M^{\nu\sigma} + \eta^{\nu\sigma}\, M^{\mu\rho} - \eta^{\nu\rho}\, M^{\mu\sigma} - \eta^{\mu\sigma}\, M^{\nu\rho} \qquad (31.4)$$

and that the $SO(8)$ generators $\tilde{M}^{ij}$ fulfill relations obtained by restricting eq. (31.4) to the i, j indices.

Consider the $SO(1,9)$ and $SO(8)$ spinorial representations. What are the dimensions of an $SO(1,9)$ and an $SO(8)$ spinors? Let us introduce the $SO(1,9)$ and $SO(8)$ Clifford algebras Γ^{μ}, γ^{i} fulfilling $\{\Gamma^{\mu},\, \Gamma^{\nu}\} = -2\,\eta^{\mu\nu}$ and $\{\gamma^{i},\, \gamma^{j}\} = -2\,\delta^{ij}$ (notice here a sign difference with respect to the definition of the $SO(8)$ gamma matrices given in the lecture notes). Show that the matrices $M^{\mu\nu} = \Gamma^{\mu\nu}/2 = (\Gamma^{\mu}\,\Gamma^{\nu} - \Gamma^{\nu}\,\Gamma^{\mu})/4$ and $\tilde{M}^{ij} = \gamma^{ij} = (\gamma^{i}\,\gamma^{j} - \gamma^{j}\,\gamma^{i})/4$ are the generators of $SO(1,9)$ and of $SO(8)$ in the spinorial representation respectively, namely that they fulfill the corresponding commutation relations. To give an explicit representation to these matrices it is useful to introduce the tensor product notation: given two matrices $A_{n\times n} = (a_{ij})$ and $B_{m\times m} = (a_{ij})$, the matrix $C_{nm\times nm} = A_{n\times n} \otimes B_{m\times m}$ is an $nm \times nm$ which is obtained by substituting each entry b_{ij} of $B_{m\times m}$ with the $n \times n$ block $b_{ij}\, A_{n\times n}$. For example consider:

$$\tau^{1} \otimes \tau^{2} \;=\; \begin{pmatrix} 0_{2\times 2} & -i\tau^{1} \\ i\tau^{1} & 0_{2\times 2} \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad (31.5)$$

where τ^{i} are the Pauli matrices. This product can be iterated by defining $A \otimes B \otimes C = A \otimes (B \otimes C)$. Show that:

$$(A_1 \otimes B_1) \cdot (A_2 \otimes B_2) \;=\; (A_1 \cdot A_2) \otimes (B_1 \cdot B_2) \qquad (31.6)$$

Show that the γ^{i} can be expressed as the tensor product of two by two matrices in the following way:

$$\begin{aligned}
&\gamma^{1} = i\tau^{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} && \gamma^{2} = i\tau^{2} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \\
&\gamma^{3} = i\tau^{3} \otimes \tau^{1} \otimes \mathbb{1} \otimes \mathbb{1} && \gamma^{4} = i\tau^{3} \otimes \tau^{2} \otimes \mathbb{1} \otimes \mathbb{1} \\
&\gamma^{5} = i\tau^{3} \otimes \tau^{3} \otimes \tau^{1} \otimes \mathbb{1} && \gamma^{6} = i\tau^{3} \otimes \tau^{3} \otimes \tau^{2} \otimes \mathbb{1} \\
&\gamma^{7} = i\tau^{3} \otimes \tau^{3} \otimes \tau^{3} \otimes \tau^{1} && \gamma^{8} = i\tau^{3} \otimes \tau^{3} \otimes \tau^{3} \otimes \tau^{2}
\end{aligned} \qquad (31.7)$$

The above representation is different from the one given in the lecture notes in which all the γ^{i} are imaginary and the $SO(8)$ generators already appear in a block diagonal form according to the decomposition into chiral representations. In the representation (31.7) however the generators which we will need to diagonalize in the sequel are diagonal to start with.

Compute the matrix $\gamma = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4\,\gamma^5\,\gamma^6\,\gamma^7\,\gamma^8$ and show that it is the chirality matrix for the $SO(8)$ spinor representation, namely that $(\gamma)^2 = 1\!\!1_{16\times 16}$ and that $\{\gamma,\,\gamma^i\} = 0$. Is γ diagonal? What are its eigenvalues? Show that the matrix $B = \tau^2\otimes\tau^1\otimes\tau^2\otimes\tau^1$ fulfills $(B)^2 = 1\!\!1_{16\times 16}$ and $B^{-1}\,\gamma^i\,B = \gamma^{i*}$. Using the results of exercise 23 and the matrices B and γ express the Majorana and the Weyl conditions on an $SO(8)$ spinor λ and show that they are compatible, so that a Majorana–Weyl spinor can be defined.

Now show that the following matrices realize the $SO(1,9)$ Clifford algebra:

$$\begin{aligned}\Gamma^0 &= \gamma\otimes\tau^2\\ \Gamma^i &= \gamma^i\otimes 1\!\!1_{2\times 2}\\ \Gamma^9 &= i\gamma\otimes\tau^1\end{aligned} \tag{31.8}$$

Compute the chirality matrix Γ. Is Γ diagonal? What are its eigenvalues? Show that the matrix $B' = \tau^2\otimes\tau^1\otimes\tau^2\otimes\tau^1\otimes\tau^3$ fulfills $(B')^2 = 1\!\!1_{32\times 32}$ and $B'^{-1}\,\Gamma^\mu\,B' = \Gamma^{\mu *}$. In terms of B' and Γ write the Majorana and the Weyl conditions on an $SO(1,9)$ spinor ξ and show that they are compatible, so that a Majorana–Weyl spinor can be defined. The group $SO(8)$ is contained inside $SO(1,9)$. The subset of the $SO(1,9)$ generators which are also $SO(8)$ generators are represented by the 32×32 matrices $M^{ij} = \Gamma^{ij}/2$. Write these generators in terms of the $SO(8)$ generators given by the 16×16 matrices $\tilde{M}^{ij} = \gamma^{ij}/2$ using the tensor product notation.

Just as for states in quantum mechanics, in order to describe a basis of the $SO(1,9)$ spinorial representation we need to define a *complete set* of $SO(1,9)$ generators, i.e. a maximal set of commuting generators, so that each element of the basis of can be labeled by the simultaneous eigenvalues of these operators. A complete set of $SO(1,9)$ generators consists of five elements $\{H_a\}$ $a = 0,\ldots,4$. Show that the following $SO(1,9)$ generators commute:

$$\begin{aligned}H_0 &= \frac{1}{2}\Gamma^0\Gamma^9\ ;\ H_1 = \frac{1}{2}\Gamma^1\Gamma^2\ ;\ H_2 = \frac{1}{2}\Gamma^3\Gamma^4\ ;\ H_3 = \frac{1}{2}\Gamma^5\Gamma^6\\ H_4 &= \frac{1}{2}\Gamma^7\Gamma^8\end{aligned} \tag{31.9}$$

Compute their expression as tensor product of two by two matrices and show that as 32×32 matrices they are diagonal. Show that H_0 has real eigenvalues, to be denoted by s_0, as opposite to $H_{k=1,2,3,4}$ which have imaginary eigenvalues, to be denoted by is_k, s_k being real. You may also convince yourself that this abelian set cannot be enlarged, i.e. that there is no other $SO(1,9)$ generator commuting with all of them. A basis for the $SO(1,9)$ spinorial representation can therefore be written in the form $\{|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle\}$, each element being a Majorana spinor, simultaneous eigenvector of the H_a:

$$H_a\,|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle = s_a\,|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle \tag{31.10}$$

Write the elements of this basis explicitly according to the various combinations of values s_a. Show that all these spinors have definite chirality and that

$$\Gamma\,|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle \;=\; 32\,s_0 s_1 s_2 s_3 s_4\,|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle \tag{31.11}$$

Let us apply the same procedure for constructing a basis of the $SO(8)$ spinorial representation. Show that a complete set of generators in $SO(8)$ is provided by the following four:

$$h_1 \;=\; \frac{1}{2}\gamma^1\gamma^2\;;\; h_2 \;=\; \frac{1}{2}\gamma^3\gamma^4\;;\; h_3 \;=\; \frac{1}{2}\gamma^5\gamma^6\;;\; h_4 \;=\; \frac{1}{2}\gamma^7\gamma^8 \tag{31.12}$$

compute the expression of h_k as tensor product of two by two matrices and show that as 16×16 matrices they are diagonal with imaginary eigenvalues is_k. A basis for the $SO(8)$ spinorial representation can therefore be written in the form of Majorana spinors $\{|s_1,\,s_2,\,s_3,\,s_4\rangle\}$. Write these basis explicitly with the various labels s_k. Show that all these spinors have definite chirality and that

$$\gamma\,|s_1,\,s_2,\,s_3,\,s_4\rangle \;=\; 16\,s_1 s_2 s_3 s_4\,|s_1,\,s_2,\,s_3,\,s_4\rangle \tag{31.13}$$

group the elements $|s_1,\,s_2,\,s_3,\,s_4\rangle$ according to their chirality and show that each chiral representation is generated by 8 elements. The representation with positive chirality is conventionally denoted by $\mathbf{8}_s$, the one with negative chirality by $\mathbf{8}_c$. Show that $H_k \;=\; h_k \otimes \mathbb{1}_{2\times 2}$ and therefore they belong to the $SO(8)$ subgroup of $SO(1,9)$. Using this property show that the spinors $|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle$ can be grouped in $SO(8)$ representations $\mathbf{8}_s$ and $\mathbf{8}_c$. For each of these representations write the eigenvalues of Γ and of H_0 (s_0).

[**Hint:** *Prove that the $SO(1,9)$ generator H_0 commutes with the $SO(8)$ subgroup of $SO(1,9)$. Therefore H_0 has a fixed eigenvalue s_0 on each irreducible representation of $SO(8)$ contained in the $SO(1,9)$ spinorial representation, by Shur's lemma.*]

Consider the vacuum of the Ramond sector $|k,\,0\rangle_R$ with momentum k^μ. It was shown to be an $SO(1,9)$ spinor, and therefore is can be expanded in the basis $|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle$. Let us implement on this state the constraint $p^\mu\,\Gamma_\mu\,|k,\,0\rangle_R \;=\; 0$. Since this state was shown to be massless ($k^\mu k_\mu \;=\; 0$) we can consider a frame in which $k^0 \;=\; k$, $k^9 \;=\; k$ and $k^i = 0$. Find the elements of the basis $|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle$ fulfilling in this frame the constraint $p^\mu\,\Gamma_\mu\,|k,\,0\rangle_R \;=\; 0$. Which $SO(8)$ representations do they belong to? Label them by the s_0 eigenvalue and the $SO(1,9)$ chirality.

[**Hint:** *Write the constraint in the chosen frame as a projector on H_0 eigenspaces:*

$$p^\mu\,\Gamma_\mu \;=\; -2\,k\,\Gamma^0\,(\frac{1}{2} - H_0)$$

]

After imposing the GSO condition on $|k,\,0\rangle_R$, namely the condition on the vacuum to have a definite $SO(1,9)$ chirality, write the spinors $|s_0,\,s_1,\,s_2,\,s_3,\,s_4\rangle$ contributing to this state for the two different eigenvalues of Γ. Which $SO(8)$ representations do they belong to?

Exercise 32

In the present exercise we shall show that the world sheet of a propagating closed string is conformally equivalent to a sphere with the asymptotic states coinciding with the two polar points.

Consider a closed string propagating in space–time. Represent its world–sheet as a cylinder parametrized by the complex coordinate $w = i\sigma_2 + \sigma_1$, where as usual σ_2 is the Euclidean time coordinate which runs from $-\infty$ (incoming string) to $+\infty$ (outgoing string) and σ_1 the angular coordinate in the interval $(0, \pi)$. Since we are dealing with a closed string the points with $\sigma_1 = 0$ and $\sigma_1 = \pi$ are identified. In the conformal gauge the metric in the coordinates w, $\bar{w}$ can be written as:

$$ds^2 = e^{\phi}\, dw\, d\bar{w} \tag{32.1}$$

write the metric in the coordinates $z = e^{2iw}$, $\bar{z}$. Represent in these coordinates the points corresponding to the asymptotic incoming and outgoing string. Now let us make a suitable choice for the conformal factor, namely:

$$e^{\phi} = 16\, \frac{|z|^2}{(1+|z|^2)^2} \tag{32.2}$$

show that the corresponding ds^2 is the metric on a sphere described by stereographic coordinates and that the asymptotic states coincide with the two polar points. [**Hint:** *Perform the change of variables* $z, \bar{z} \to \theta, \varphi$ *where* $z = \cot(\frac{\theta}{2})\, e^{i\varphi}$]

In general, for a more complicated process involving several incoming and outgoing strings, it can be shown that, at tree level (that is if the surface described by the interacting strings has no holes), the conformal factor of the metric can be chosen so as to map the surface of the initial diagram into a sphere in which the asymptotic states are represented by points or *punctures*, where the corresponding vertex operators are inserted. Analogously higher loop diagrams will be mapped into two dimensional surfaces with a certain number of holes (counted by the genus g of the surface) and punctures corresponding to the asymptotic states.

Exercise 33

Consider type IIA and IIB superstring theories. Write the explicit state realization of the NS–NS massless modes, namely of the scalar field ϕ, the ten dimensional metric G_{ij} and the two form B_{ij} (also called Kalb–Ramond field). As far as the R–R sector is concerned, the zero mode states are expressed as tensor product of the left and right mover sector ground states (we shall use the notation of the lecture notes and denote by $|\psi_L\rangle$, $|\psi_R\rangle$ the R ground states with positive and negative $SO(8)$ chirality respectively (which we have called $\mathbf{8}_s$, $\mathbf{8}_c$ in exercise 31). The states corresponding to the left or right moving sector are distinguished only by their positions (to the left or to the right) in the tensor products). For type IIA and IIB the zero modes

of the R–R sector are $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$. These states have to be considered as 8×8 matrices in the spinor space and therefore can be decomposed in a complete basis of generators of this space, namely $\Gamma^{i_1 i_2\ldots i_k}$ for $k = 1,\ldots,8$. Show that only the matrices $\Gamma^{i_1 i_2\ldots i_k}$ for $k = 1,\ldots,4$ contribute to the expansions of $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$.
As a first exercise let us consider the simpler case of bi–spinors ξ^α, χ^β. Show that, if we denote by $M^{\alpha\beta} = \xi^\alpha\,\chi^\beta$, the following expansion is true:

$$\begin{aligned} M &= \sum_i c_i\,\mathrm{Tr}(M^T\,\tau^i)\,\tau^i \\ c_i &= \frac{1}{\mathrm{Tr}(\tau^{iT}\,\tau^i)} \end{aligned} \tag{33.1}$$

where τ^i are the Pauli matrices and $\mathrm{Tr}(M^T\,\tau^i) = \xi^T\,\tau^i\,\chi$.
Similarly we can write:

$$\begin{aligned} |\psi_L\rangle\langle\psi_R| &= \sum_k c_k\,C^A_{i_1\ldots i_k}\,\Gamma^{i_1 i_2\ldots i_k} \\ |\psi_L\rangle\langle\psi_L| &= \sum_k c_k\,C^B_{i_1\ldots i_k}\,\Gamma^{i_1 i_2\ldots i_k} \\ C^A_{i_1\ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2\ldots i_k}|\psi_R\rangle \\ C^B_{i_1\ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2\ldots i_k}|\psi_L\rangle \end{aligned} \tag{33.2}$$

where c_k are numerical constants. Show that the fields $C^A_{i_1\ldots i_k}$ are non vanishing only for $k = 1$, 3 and $C^B_{i_1\ldots i_k}$ are non vanishing only for $k = 0$, 2, 4. Which $SO(8)$ representations do the C^A and C^B tensors belong to?
Write the explicit state realization of the NS–R and R–NS massless modes. Perform the counting of fermionic and bosonic degrees of freedom.

Exercise 34

In the present exercise we shall study the unoriented closed superstring theory. Unoriented theories are invariant under the transformation $\Omega\,:\,\sigma\to\pi-\sigma$. What is the effect of this transformation on the left and right mode operators? Show that only type IIB theory can be made invariant under Ω. Show that the NS–NS fields ϕ, G_{ij} are even under Ω and that B_{ij} is odd [**Hint:** *Write the corresponding states as components of* $b^i_{-1/2}\,\tilde{b}^j_{-1/2}\,|0,0\rangle$ *and show that the action of* Ω *amounts to* $i\leftrightarrow j$.]
As far as the R–R sector is concerned, show that under Ω C_{ij} is even and C_{ijkl} is odd. As far as the fermion fields are concerned show that the effect of Ω is to switch NS-R $\leftrightarrow$ R–NS and therefore only a symmetric combination of the two survives (is even). Perform a counting of the fermionic and bosonic degrees of freedom and motivate the fact that the amount of supersymmetry is $\mathcal{N} = 1$.