Rational cohomology of an algebra need not be detected by Frobenius kernels

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Abstract

We record some cohomological computations in an example of Friedrich Knop. The example is a polynomial algebra in characteristic two with an unusual SL_2 action.

1 The example

In [4, section 5] Friedrich Knop gave an example of a transitive action of the algebraic group SL_2 on the afine plane $\mathbb{A}^2$ in characteristic two. At the 2002 CRM Workshop on Invariant Theory, cf. [5], he also explained that the example has noteworthy properties in connection with Grosshans grading. In this note we look at the rational cohomology and see that it is equally instructive. Thus consider the algebraic group $G = \mathrm{SL}_2$ defined over a field k of characteristic two. The diagonal subgroup is T, the unipotent upper triangular subgroup is U. Let N be the normalizer of T in G. The example is then $G/N \cong \mathbb{A}^2$. As Knop observed, $k[G/N] \hookrightarrow k[G/T]$ is separable, $k[G/T]$ has good filtration and $k[G/N]^U = k[G/T]^U$. Recall from [5, 2.3] that the Grosshans graded $\mathrm{gr}\, k[G/T]$, known as the 'hull' of $\mathrm{gr}\, k[G/N]$, is a purely inseparable extension of $\mathrm{gr}\, k[G/N]$. So in some sense the inseparability is a property of the Grosshans filtration, not of the ring extension. We will give cohomology computations that amplify these observations.

We write the general matrix in G as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that

$$k[G] = k[a, b, c, d]/(ad - bc - 1).$$

Recall that $k[U\backslash G] = k[c, d]$, a polynomial ring in two variables. Here by $k[U\backslash G]$ we mean the ring of rational functions on G that are invariant under left translation by elements of U. (One may check $k[U\backslash G] = k[c, d]$ using the multiplicities in a good filtration of $k[G]$ as $G \times G$ module.) Thus $k[G/T]^U = k[U\backslash G]^T = k[cd] = k[U\backslash G]^N = k[G/N]^U$, a polynomial ring in one variable, written cd to indicate its image in $k[G]$. Note that $ad \in k[G/T]$ satifies $ad(ad-1) = abcd \in k[G/N]$, while no power $(ad)^{2^r}$ is in $k[G/N]$, because the involution $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N$ interchanges ad with bc, but $(ad)^{2^r} - (bc)^{2^r} = 1$. It is easy to see that $k[G/T]$ is generated by ad, ab, cd. Using the involution σ again, one sees that $k[G/T]$ is a free $k[G/N]$-module with basis 1, ad. Thus $k[G/N] \hookrightarrow k[G/T]$ is separable, $x^2 - x - (ab)(cd)$ being the minimal polynomial of ad. One also finds that $k[G/N]$ is a polynomial ring in two variables, called ab and cd. One may find a basis of $k[G/T]$ and make its (good) Grosshans filtration explicit as a check of the above.

The Grosshans filtration of $k[G/T]$ starts with the span of 1, then the span of 1, ab, ad, cd. Intersecting with $k[G/N]$ one gets the span of 1 and the span of 1, ab, cd respectively. So the class of ad is an invariant in $k[G/T]/k[G/N]$ and this defines an extension of k by $k[G/N]$ that does not split because $k[G/N]^G = k[G/T]^G = k$. But $ad - (ad)^{2^r} \in k[G/N]$ for all $r \geq 1$, so for every such r the extension splits as an extension of modules for the Frobenius kernel G_r, with $(ad)^{2^r} \in k[G/T]^{G_r}$ as lift of the class of ad. We have shown:

Proposition 1 *$H^1(G, k[G/N])$ is not the inverse limit of the $H^1(G_r, k[G/N])$.*

This is in contrast with what one knows for finite dimensional representations [3, II 4.12].

One may go a little further and compute both $H^*(G, k[G/N])$ and $H^*(G_r, k[G/N])$. As $k[G/T]/k[G/N]$ is a free $k[G/N]$ module on one generator, we get an the extension

$$\mathcal{E}: \quad 0 \to k[G/N] \to k[G/T] \to k[G/N] \to 0.$$

Further $k[G/T]$ is a direct summand of the injective module $k[G]$, so it is easy to compute that $H^i(G, k[G/N]) = k$ for all $i \geq 0$. More specifically, if f denotes the inclusion $k \hookrightarrow k[G/N]$, then $H^i(G, k[G/N]) = k$ is spanned for $i \geq 1$ by the Yoneda product of f and i copies of $\mathcal{E}$. So, by [2, 3.2], $H^*(G, k[G/N])$ is a polynomial ring in one variable of degree one.

Let $r \geq 1$. Now $k[G]$ is also G_r injective, by [3, I 4.12, 5.13]. But $\mathcal{E}$ splits over G_r, so one gets $H^i(G_r, k[G/N]) = 0$ for $i > 0$. So the G_r

detect none of the higher cohomology of $k[G/N]$. One may also check that $H^0(G_r, k[G/N])^{(-r)} \cong k[G/N]$. Of course $H^0(G, k[G/N]) = k$ is the inverse limit of the $H^0(G_r, k[G/N])$.

We leave it to the reader to compute the nontrivial $H^*(G_1, \mathrm{gr}\ k[G/N])$. (Use [1, 3.10].)

References

[1] H. H. Andersen, J.-C. Jantzen, Cohomology of induced representations for algebraic groups, Math. Ann. 269 (1984), 487–525.

[2] D. J. Benson, Representations and cohomology. I. Basic representation theory of finite groups and associative algebras. Second edition. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge, 1998.

[3] Jens Carsten Jantzen, Representations of algebraic groups. Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003.

[4] Friedrich Knop, Homogeneous varieties for semisimple groups of rank one. Compositio Mathematica, 98 (1995), 77–89.

[5] Wilberd van der Kallen, Cohomology with Grosshans graded coefficients, In: Invariant Theory in All Characteristics, Edited by: H. E. A. Eddy Campbell and David L. Wehlau, CRM Proceedings and Lecture Notes, Volume 35 (2004) 127-138, Amer. Math. Soc., Providence, RI, 2004.