HOMOLOGY STABILITY FOR THE GENERAL LINEAR GROUP

PROEFSCHRIFT

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Wilt heden nu treden, voor God den Here,

Hem boven al loven van herten zeer.

En maken groot Zijns lieven namens ere.

....

Maakt U, o mens, voor God steeds wel te dragen,

Doet ieder recht en wacht u voor bedrog.

Adriaan Valerius.

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Introduction.

For a given ring R, the groups $GL_n(R)$ form a direct system in the sense that $GL_n(R)$ sits embedded in $GL_{n+1}(R)$ when one sends $A \in GL_n(R)$ to $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$. We consider the following problem: prove that the integral homology of $GL_n(R)$ stabilizes in the sense that for given i, the homomorphism $H_i(GL_n(R),\mathbb{Z}) \to$ $\to H_i(GL_{n+1}(R),\mathbb{Z})$ is an isomorphism when n is large enough with respect to i. A known but unpublished result of Quillen states: if K is a field different from $\mathbb{F}_2$ then $H_i(GL_n(K),\mathbb{Z}) \to$ $\to H_i(GL_{n+1}(K),\mathbb{Z})$ is an isomorphism for $n > i$.

The symmetric groups S_n form a direct system just as $GL_n(R)$. We shall describe a method to prove Nakaoka's theorem on the homology stability for S_n. We shall generalise this method to get homology stability for $GL_n(R)$ for a class of rings including local rings.

Assume S_n acts on a reasonable topological space F. Construct a fibre space X with fibre F over the classifying space BS_n of S_n in the following way. Let $ES_n \to BS_n$ be a principal fibration. The group S_n acts on the right on ES_n in such a way that $ES_n/S_n = BS_n$. Form $ES_n \times F$ and divide out the action of S_n given by $g(e,f) = (eg^{-1},gf)$ for $g \in S_n$, $e \in ES_n$, $f \in F$. The resulting quotient space is X.

Projection onto the first factor yields a map $X \to BS_n$, which is a fibration with fibre F. If F is acyclic in low dimensions, the homology spectral sequence yields that $H_i(X,\mathbb{Z}) \to H_i(S_n,\mathbb{Z})$ is an isomorphism for small i.

Projection onto the second factor yields a map $X \to S_n\backslash F$

The fibre above the image of an $f \in F$ in $S_n \backslash F$ is known to be $BStab_{S_n}(f)$. So X can be conceived of as consisting of spaces BH with H a subgroup of S_n, glued together in some way. If $S_n \backslash F$ is reasonable and the homology of H is independent of H in low dimensions, we expect that there is a subgroup G_0 of S_n such that $H_i(X,\mathbf{Z})$ is equal to $H_i(G_0,\mathbf{Z})$ for small i.

Now it is clear what our program should be: construct an F which is acyclic in low dimensions such that BS_{n-1} sits in X, and the inclusion $BS_{n-1} \to X$ induces an isomorphism on homology in low dimensions.

The group S_n acts on the topological n-1-simplex Δ^{n-1} by permuting the vertices. The boundary of Δ^{n-1} is the n-2-sphere. However, the barycentres of the simplices of S^{n-2} have stabilizers which are too large and too complicated. To avoid this, we build a topological space made of n! copies of Δ^{n-1}, one for each ordering of 1,...,n. We might describe this space as the realisation of a certain semi-simplicial set, but we gain by passing to barycentric subdivision. The space then becomes the realisation of some partially ordered set (abbreviated poset), and a large machinery to handle these is available.

In the context of barycentric subdivision, the n-1-simplex is the realisation of the poset of all non-empty subsets of {1,...,n}. In our new space, we want to distinguish between (1,2) and (2,1), so we define

$$\mathcal{O}(n) = \{(i_1,\ldots,i_k) \mid 1 \leq i_j \leq n,\ i_s \neq i_t \text{ if } s \neq t\}$$

and we let $(i_1,\ldots,i_k) \leq (j_1,\ldots,j_l)$, if the latter sequence is a refinement of the first. We prove by induction on n that the realisation $|\mathcal{O}(n)|$ of $\mathcal{O}(n)$ is n-1-spherical, by observing first that $|\mathcal{O}(n-1)|$ sits in $|\mathcal{O}(n)|$, and then glueing the remaining part of $|\mathcal{O}(n)|$ to it, killing the homology in dimension n-2.

To carry out the rest of our program for the symmetric groups, we use the homology theory of categories, because it is a flexible apparatus. If $\mathcal{C}$ is a category, then the categorical homology $H_*(\mathcal{C},\mathbb{Z})$ is naturally isomorphic to the homology $H_*(|\mathcal{C}|,\mathbb{Z})$ of the realisation $|\mathcal{C}|$ of $\mathcal{C}$. Hence we can switch between both homology theories at will.

The stabilizer of $(n) \in |\mathcal{O}(n)|$ in S_n is equal to S_{n-1}. We manufacture a category $\mathcal{X}$ such that $|\mathcal{X}| = X$ is the fibre space we were after. With the help of homology of categories, we show that $BS_{n-1} \to X$ induces an isomorphism on homology in low dimensions, the main tool being a spectral sequence for a functor $f : \mathcal{C} \to \mathcal{C}'$ connecting the homologies of $\mathcal{C}$ and $\mathcal{C}'$. This finishes the program for the symmetric groups.

Let R be a ring. To prove homology stability for $GL_n(R)$, we make it act on

$\mathcal{O}(n,R)=\{(v_1,\ldots,v_p)\,|\,v_1,\ldots,v_p \in R^n, \exists v_{p+1},\ldots,v_n \ \det(v_1,\ldots,v_n)\in R^*\}$

If R is a local ring or a Euclidean ring, we prove by a more complicated version of the same scissors-and glue argument that $\mathcal{O}(n,R)$ is n-1-spherical. However, if we proceed as in the case of the symmetric groups, a problem arises:

$\mathrm{Stab}_{GL_n(R)}(1,0,\ldots,0) \cong GA_{n-1}(R)$, the semi-direct product of $GL_{n-1}(R)$ and R^{n-1}, so this stabilizer is not $GL_{n-1}(R)$ but a little larger. Therefore we would like to know that

$H_i(GL_{n-1}(R),\mathbb{Z}) \to H_i(GA_{n-1}(R),\mathbb{Z})$ is an isomorphism for small i.

To prove the latter statement, we make $GA_n(R)$ act on the affine analogue of $\mathcal{O}(n,R)$, viz.

$$\mathcal{A}(n,R)=\{(v_0,\ldots,v_p)\,|\,v_0,\ldots,v_p\in R^n,(v_1-v_0,\ldots,v_p-v_0)\in\mathcal{O}(n,R)\}$$

Observe that $Stab_{GA_n(R)}(0) = GL_n(R)$. Using scissors-and-glue, we can show that $\mathcal{A}(n,R)$ is n-spherical if R is local. However, this proof does not generalise to give the desired acyclicity for other rings. So we present another proof, using induction on n in the following way: we prove that $\tilde{H}_i(\mathcal{A}(n,R),\mathbb{Z}) = 0$ for $i < n-1$ and then, by way of a careful comparison with the homology of a Tits building, we calculate explicit generators for the group $\tilde{H}_{n-1}(\mathcal{A}(n,R),\mathbb{Z})$. The proof draws on a discussion of so-called homogeneousmorphisms between homogeneous posets. We finally employ (simple) arithmetical arguments to show that these generators are zero if R is a field or a subring of $\mathbb{Q}$. Thus we find that $\mathcal{A}(n,R)$ is n-spherical if R is local or a subring of $\mathbb{Q}$.

By means of these acyclicity results we prove now, in the same vein as for the symmetric groups, for R a local ring or a subring of $\mathbb{Q}$ that $H_i(GL_n(R),\mathbb{Z}) \to H_i(GL_{n+1}(R),\mathbb{Z})$ is an isomorphism for $n > 2i$ and surjective for $n = 2i$.

Results about homology stability for $GL_n(R)$ with R a Dedekind domain were obtained independently by R. Charney [2], using a different approach.

Table of contents

List of conventions and notations.

All rings are commutative with unit. A (topological) space is always a CW complex. The sign ⊠ means end of proof.

References are given with the number of the chapter in Roman numerals and the number of the item; references inside the same chapter drop the chapter number.

We give a list of principal notations. It has no pretention of completeness what so ever.

Notation	Meaning	See section
$H_i(X,A)$	the i-th homology group of the topological space X with coefficients in A.	
$\tilde{H}_i(X,A)$	the i-th reduced homology group of X with coefficients in A, i.e. the homology of the augmented chain complex.	
Hopf(X,Y)	The join of the spaces X and Y	I1
CX	cone over the space X	I1
S^k	k-sphere	I1
SX	suspension of the space X	I1
S^kX	k-fold suspension of X	I1
$N_*\mathcal{C}$	nerve of a small category $\mathcal{C}$, it is a semi-simplicial set	I2
Δ^k	k-simplex	I2
$B\mathcal{C}$, $\|\mathcal{C}\|$	realisation of the small category $\mathcal{C}$	I2
f/Y	$f : \mathcal{C} \to \mathcal{C}'$ a functor, $Y \in Obj(\mathcal{C}')$, see	I2
$Y\backslash f$	see	I2

$\mathcal{C}_*(\mathcal{C},\mathcal{L})$	chains of the category $\mathcal{C}$ with values in the system of coefficients $\mathcal{L}$	I3
$H_*(X,\mathcal{L})$	$H_*(\mathcal{C}_*(X,\mathcal{L}))$	I3
$\mathcal{P}(X,L)$	a certain acyclic system of coefficients on the category $\mathcal{C}$, defined for $X \in \mathrm{Obj}\,\mathcal{C}$ L an abelian group	I3
$X * Y$	poset constructed from posets X and Y, see definition (1.1)	II1
$H(X,Y)$	the same	II1
$\mathrm{Link}_Y(x)$	if Y is a subposet of X, then $\mathrm{Link}_Y(x)=\{y \in Y \mid y < x \text{ or } y > x\}$	II1
$\mathrm{Link}_Y^-(x)$	part of $\mathrm{Link}_Y(x)$	II1
$\mathrm{Link}_Y^+(x)$	other part of $\mathrm{Link}_Y(x)$	II1
$ht_X(x)$	the height of the element x of the poset X	II2
dX	maximal chain length in the poset X	II2
$X_{\leqslant k}$	elements of height $\leqslant k$ in the poset X	II2
$X_{\geqslant k}$	same with $\geqslant k$	II2
$\mathcal{O}(A_*)$	ordered set of non-degenerate simplices of the semi-simplicial set A_*	II4
$\mathcal{O}(V)$	If V is a set, $\mathcal{O}(V)$ is the poset of all sequences of distinct elements in V, ordered in a natural way.	II4
$N_*^!F$	semi-simplicial set associated to $F \subseteq \mathcal{O}(V)$ having the chain property	II4
X_p	p-skeleton of the CW complex X	II4

$O(n)$	$\mathcal{O}(\{1,\ldots,n\})$	II4
$\mathcal{O}(n,R)$	R a ring, see examples in	II4
$\mathcal{A}(n,R)$	R a ring, see examples in	II4
$F_{(v_0,\ldots,v_p)}$	if $F \subseteq \mathcal{O}(V)$ is a full subposet, then $F_{(v_0,\ldots,v_p)}$ is a subposet of F which is again a full subposet of $\mathcal{O}(V)$	II5
Z_nF	if $F \subseteq \mathcal{O}(V)$ is full, then Z_nF is the poset of all sequences in F with new elements $z_0,\ldots,z_n$ inserted in that order	II5
$F < S >$	if $F \subseteq \mathcal{O}(V)$ is full and S a non-empty set then $F < S >$ is a full subposet of $\mathcal{O}(V \times S)$	II6
$\mathcal{O}(n,k,R)$	if $e_1,\ldots,e_n$ is the standard basis in R^n then $\mathcal{O}(n,k,R) = \mathcal{O}(n,R)_{(e_1,\ldots,e_k)}$ for $k < n$	III3,4
$T(n,R)$	Tits building of R^n	III5
$A(n,R)$	its affine analogue	III5
$g(L_1,\ldots,L_n)$	generator for $\tilde{H}_{n-2}(T(n,R),\mathbb{Z})$, R a Euclidean ring, see (5.2)	III5
$g(v_0,\ldots,v_n)$	the same for $H_{n-1}(A(n,R),\mathbb{Z})$	III5
$\mathcal{A}(n,k,R)$	see definition (5.8)	III5
$GL_n(R)$	group of invertible $n \times n$ matrices over R, the general linear group	
$GA_n(R)$	general affine group	IV1
$\mathcal{X}(F,G)$	if G is a group acting transitively on the full $F \subseteq \mathcal{O}(V)$ then $\mathcal{X}(F,G)$ is a category with $Obj(\mathcal{X}(F,G)) = F$;	IV1

	for $\vec{v},\vec{w} \in F$ the set $\mathrm{Mor}(\vec{v},\vec{w}) = \{g \in G \mid g\vec{v} \leq \vec{w}\}$	
$\pi(F,G)$	projection functor $\mathcal{X}(F,G) \to G$	IV1
$Q(n)$	a certain subcategory of the category of ordered sets	IV2
δ_i^k	morphisms in $Q(n)$, defined for $1 \leq k \leq n$, $0 \leq i \leq k$	IV2
$C_*^{red}(Q(n)^0,\mathcal{L})$	a complex such that $H_*(C_*^{red}(Q(n)^0,\mathcal{L}))=H_*(Q(n)^0,\mathcal{L})$, see IV (2.1)	IV2
$\rho(F,G)$	projection functor $\mathcal{X}(F,G) \to Q(dF+1)$	IV2
$E_2^{pq}(F,G)$	E^2-term of spectral sequence of $\rho(F,G)$	IV2
$(e_1,\ldots,e_{d+1})$	see	IV2
G_i	see	IV2
$\mathcal{C}_p^{(q)}$	see	IV2
$\partial_p^{(q)}$	see	IV2
$GA_n^k(R)$	$GL_n(R) \ltimes (R^n)^k$	IV3
(α_m), (β_m)	statements about the homology of $GL_n(R)$ and $GA_n^k(R)$, see IV (3.2)	IV3
$GA_n^{k,l}(R)$	$GA_n^k(R) \ltimes (R^{n+k})^l$	IV4
$\tau_n^{k,l}$	isomorphism between $GA_n^{k,l}(R)$ and $GA_n^{l,k}(R)$	IV4
(γ_m)	statement about the homology of $GA_n^{k,l}(R)$ and $GL_n(R)$ implying (β_m)	IV4

I. Preliminaries.

1. Some topological remarks.

A topological space will always mean a C.W.-complex. For technical reasons, we'll always endow it with the k-topology in the sense of Steenrod [14]. We'll call it a space, sometimes.

The reader is assumed to be familiar with the elementary notions of algebraic topology, such as Mayer-Vietoris sequences, etc.

Because it will play an important role, we shall give a brief discussion of the join or Hopf-construction here. See Milnor [5] for more details.

Intuitively, for two spaces X,Y, the space Hopf(X,Y) consists of all line segments joining a point of X to a point of Y. Defining the cone over X by

$$CX = X \times [0,1]/X \times \{0\}$$

one sees one can define

$$\mathrm{Hopf}(X,Y) = CX \times Y \cup_{X\times Y} X \times CY .$$

Recall that $X/\phi = X \cup \{pt\}$ so that $\mathrm{Hopf}(X,\phi) = X$. The Hopf-construction is commutative, and associative in the following sense:

(1.1) $\quad \mathrm{Hopf}(X,\mathrm{Hopf}(Y,Z)) = \mathrm{Hopf}(\mathrm{Hopf}(X,Y),Z)$

Taking S^k the k-sphere, one has

(1.2) $\quad \mathrm{Hopf}(X,S^0) = SX = CX \cup_X CX$

the suspension of X. Since

$$S^k = SS^{k-1} = \mathrm{Hopf}(S^{k-1},S^0)$$

one sees by (1.1) that

(1.3) · $\mathrm{Hopf}(X,S^k) = S^{k+1}X$

the k+1-fold suspension of X.

The definition of Hopf(X,Y) allows us to write down a Mayer-Vietoris sequence connecting its homology with coefficients in a commutative ring A to the homology of X and Y. We have in fact

$$(1.4) \ldots \to H_i(X \times Y,A) \to H_i(X,A) \oplus H_i(Y,A) \to H_i(\mathrm{Hopf}(X,Y),A) \to$$
$$\to H_{i-1}(X \times Y,A) \to \ldots$$

We say that X has the homology of a wedge of k-spheres over A or is a homology-wedge of k-spheres over A if the reduced homology of X with coefficients in A has the following properties:

$$\tilde{H}_i(X,A) = 0 \text{ if } i \neq k$$

$\tilde{H}_k(X,A)$ is free over A.

Note that the empty set is a homology-wedge of (-1)-spheres over A because $\tilde{H}_{-1}(\phi,A) = A$, $\tilde{H}_i(\phi,A) = 0$ for $i \geq 0$.

Assume now X,Y are homology-wedges of k,l-spheres over A then by Künneth we have

$$H_n(X \times Y,A) = \bigoplus_{i+j=n} H_i(X,A) \otimes H_j(Y,A) .$$

If k,l = 0 we find

$$H_0(X \times Y,A) = H_0(X,A) \otimes H_0(Y,A)$$

and by (1.4)

$$H_1(\mathrm{Hopf}(X,Y),A) = \tilde{H}_0(X,A) \otimes \tilde{H}_0(Y,A) .$$

In case k = 0, l > 0 we have

$$H_0(X \times Y,A) = H_0(X,A)$$
$$H_l(X \times Y,A) = H_l(Y,A) \otimes H_0(X,A)$$

so by (1.4)

$$H_{l+1}(\mathrm{Hopf}(X,Y),A) = H_l(Y,A) \otimes \tilde{H}_0(X,A).$$

And if $k,l > 0$ we find

$$H_{k+l+1}(\mathrm{Hopf}(X,Y),A) = H_k(X,A) \otimes H_l(Y,A) .$$

In all cases, if $i \neq k+l+1$ we find

$$H_i(\mathrm{Hopf}(X,Y),A) = 0.$$

As $\mathrm{Hopf}(\phi,X) = X$, we have shown for all $k,l \geq -1$:

(1.5) Proposition. If X,Y are homology-wedges of k,l-spheres over A, then $\mathrm{Hopf}(X,Y)$ is a homology wedge of $k+l+1$-spheres over A, and

$$\tilde{H}_{k+l+1}(\mathrm{Hopf}(X,Y),A) \cong \tilde{H}_k(X,A) \otimes \tilde{H}_l(Y,A) .$$

2. The realisation of a small category.

Let $\mathcal{C}$ be a small category, i.e. a category whose objects form a set. Its realization can be viewed as a geometric "picture" of $\mathcal{C}$, in which objects are represented by points, morphisms by line segments between appropriate points, commutative triangles of morphisms by solid triangles and so on.

In this section, we shall give the definition of the realisation of a small category and we'll recall some properties. Furthermore we introduce some technical notions. For more details, see Segal [12], Quillen [8], §1.

Let $\mathcal{C}$ be a small category. Its nerve $N_*\mathcal{C}$ is a semi-simplicial set, defined as follows:

$$N_k\mathcal{C} = \{X_0 \xrightarrow{f_1} X_1 \to \dots \xrightarrow{f_p} X_p \mid X_i \in \mathrm{Obj}(\mathcal{C}),\ f_i \in \mathrm{Mor}_{\mathcal{C}}(X_{i-1},X_i)\}$$

Let $\partial_i : N_k\mathcal{C} \to N_{k-1}\mathcal{C}$ be given by deleting X_i, and $\sigma_i : N_k\mathcal{C} \to$ $\to N_{k+1}\mathcal{C}$ by replacing X_i by $X_i \xrightarrow{id_{X_i}} X_i$.

The standard k-simplex Δ^k is the convex closure in $\mathbb{R}^k$ of k+1 points in general position. A point of Δ^k is then represented by a sequence $(t_0,..,t_k)$, $t_i \in [0,1]$, $\sum_{i=0}^{n} t_i = 1$, by using barycentric coordinates. Define for $i = 0,\dots,k+1$

$$\delta_i : \Delta^k \to \Delta^{k+1}$$

$$(t_0,\dots,t_k) \mapsto (t_0,\dots,t_{i-1},0,t_i,\dots,t_k)$$

and for $i = 0,\dots,k-1$

$$s_i : \Delta^k \to \Delta^{k-1}$$

$$(t_0,\dots,t_k) \mapsto (t_0,\dots,t_i+t_{i+1},\dots,t_k)$$

Now form

$$\coprod N_k C \times \Delta^k$$

and divide out the equivalence generated by

$$(\sigma_i x,t) \sim (x,s_i t)$$

$$(\partial_i x,t) \sim (x,\delta_i t) .$$

The resulting quotient space, endowed with the k-topology, is the (geometric) realisation of C, denoted $B\mathcal{C}$ or $|\mathcal{C}|$. If $f : \mathcal{C} \to \mathcal{C}'$ is a functor between small categories, then $N_*f : N_*\mathcal{C} \to N_*\mathcal{C}'$, defined by

$$N_k(f)(X_0 \xrightarrow{f_1} \dots \xrightarrow{f_k} X_k) = fX_0 \xrightarrow{f(f_1)} \dots \xrightarrow{f(f_k)} fX_k$$

is a morphism of semisimplicial sets. It gives rise to a continuous map $Bf : B\mathcal{C} \to B\mathcal{C}'$.

The realisation has the following properties.

(2.1) <u>Proposition</u>. (i) If $\mathcal{C},\mathcal{C}'$ are small categories, then $B(\mathcal{C} \times \mathcal{C}') = B\mathcal{C} \times B\mathcal{C}'$, and if $\mathcal{C}^0$ is the opposite category of $\mathcal{C}$ then $B\mathcal{C} = B\mathcal{C}^0$.

(ii) A natural transformation of functors $f,g : \mathcal{C} \to \mathcal{C}'$ induces a homotopy between Bf and Bg.

(iii) If a functor f has a (left or right) adjoint then Bf is is a homotopy equivalence.

(iv) A category having an initial or final object has contractible realisation.

Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor of small categories. To describe f the categories f/Y are useful. They are defined as follows:

$$\mathrm{Obj}(f/Y) = \{(X,v) \mid X \in \mathrm{Obj}(\mathcal{C}),\ v : f(X) \to Y\} .$$

If (X,v) and (X',v') are in $\mathrm{Obj}(f/Y)$, a morphism $w : (X,v) \to (X',v')$ is a $w : X \to X'$, such that $v'f(w) = v$. In what follows $X \in \mathrm{Obj}\,\mathcal{C}$ and $v \in \mathrm{Mor}\,\mathcal{C}$ will be abbreviated to $X \in \mathcal{C}$, $v \in \mathcal{C}$ respectively.

Denote by $f^{-1}(Y)$ the fibre of Y, i.e. the subcategory of $\mathcal{C}$ consisting of objects mapped to Y and morphisms mapped to id_Y. We obviously have a functor $f^{-1}(Y) \to f/Y$, sending $X \mapsto (X, id_Y)$.

Suppose now these functors have left adjoints $(X,v) \mapsto v_*X$. If $v : Y \to Y'$ is a morphism, then for $X \in f^{-1}Y$, $X \mapsto (X,v) \mapsto v_*X$ is a functor $v_* : f^{-1}(Y) \to f^{-1}(Y')$ called cobase change. We call such a functor f precofibred.

If $v : Y \to Y'$, $u : Y' \to Y''$ are morphisms, then there is a morphism of functors $(uv)_* \to u_*v_*$ coming from the morphism $(X,v) \to (v_*X, id_{Y'})$. If for all composable u,v, $(uv)_* \to u_*v_*$ is an isomorphism, we call f cofibred.

Reversing all arrows, we get the dual notions of $Y\backslash f$, prefibred and fibred.

For a functor $f : \mathcal{C} \to \mathcal{C}'$ to be a homotopy equivalence, one has the following "theorem A" of Quillen [8].

(2.2) Theorem. If $f : \mathcal{C} \to \mathcal{C}'$ is a functor of small categories and $B(f/Y)$ is contractible for all $Y \in \mathcal{C}'$ then Bf is a homotopy equivalence. The same holds if all $B(Y\backslash f)$ are contractible.

3. Homology of categories.

If $\mathcal{C}$ is a small category, then a system of coefficients on $\mathcal{C}$ is a functor $\mathcal{L} : \mathcal{C} \to \underline{Ab}$, $\underline{Ab}$ being the category of abelian groups. We shall define homology groups of $\mathcal{C}$ with coefficients in $\mathcal{L}$ which are seen to be closely related to the homology of the topological space $B\mathcal{C}$. For more details and proofs of the results in this section, the reader is referred to Gabriel-Zisman [3] app. II.

The systems of coefficients on a small category $\mathcal{C}$ form an abelian category. If $\mathcal{L}$ is such a system, define

$$\mathcal{C}_p(\mathcal{C},\mathcal{L}) = \coprod_{X_0 \to \dots \to X_p} \mathcal{L}(X_0)$$

Now $\mathcal{C}_*(\mathcal{C},\mathcal{L})$ is a simplicial abelinan group, with obvious definitions of boundaries and degeneracies. Its homology groups are called the homology groups of $\mathcal{C}$ with coefficients in $\mathcal{L}$ and denoted $H_p(\mathcal{C},\mathcal{L})$.

A system of coefficients on $\mathcal{C}$ is called morphism-inverting if $\mathcal{L}(f)$ is an isomorphism for each $f \in \mathcal{C}$. We call $\mathcal{L}$ constant if $\mathcal{L}(f) = \mathrm{id}$ for all $f \in \mathcal{C}$.

A morphism-inverting system of coefficients on $\mathcal{C}$ gives rise to a local system of coefficients on $B\mathcal{C}$. Denoting both by $\mathcal{L}$ we have by Quillen [8] §1

$$H_p(B\mathcal{C},\mathcal{L}) = H_p(\mathcal{C},\mathcal{L}) .$$

Example. Let G be a group. We can view it as a category with one object, also denoted by G. A functor $\mathcal{L} : G \to \underline{Ab}$ is just an abelian G-group L. We have $H_p(G,L) = H_p(G,\mathcal{L})$, i.e. group homology coincides with categorical homology, as we see by comparing complexes.

Furthermore $\mathcal{L}$ is always morphism-inverting, so $H_p(G,L) = H_p(G,\mathcal{L}) = H_p(BG,\mathcal{L})$. Recall that BG is a K(G,1)-space in the sense of Eilenberg-MacLane.

(3.1) Proposition. Let $\mathcal{L}$ be a system of coefficients on the small category $\mathcal{C}$. The group $H_0(\mathcal{C},\mathcal{L})$ can be identified with $\varinjlim^{\mathcal{C}}\mathcal{L}$, and $H_n(\mathcal{C},\mathcal{L})$ can be identified with the n-th left satellite of $\varinjlim^{\mathcal{C}}$, so

$$H_n(\mathcal{C},\mathcal{L}) = \varinjlim{}_n^{\mathcal{C}}\, \mathcal{L}.$$

The proof of this result uses the existence of acyclic coverings. Because we need the precise form of an acyclic covering later on, we'll give a proof of this here. Recall that a system of coefficients $\mathcal{P}$ is called acyclic if $H_i(\mathcal{C},\mathcal{P}) = 0$, for $i > 0$.

(3.2) Proposition. If $\mathcal{L}$ is a system of coefficients on a small category $\mathcal{C}$, then there is an acyclic system $\mathcal{P}$ such that $\mathcal{P} \to \mathcal{L} \to 0$ is exact.

Proof. Let $X \in \mathcal{C}$ and denote $X\backslash\mathcal{C} = X\backslash id_{\mathcal{C}}$. If L is an abelian group, let $L_{X\backslash\mathcal{C}}$ be the constant system on $X\backslash\mathcal{C}$ with value L. Let $p : X\backslash\mathcal{C} \to \mathcal{C}$ be the projection functor and define

$$\mathcal{P}(X,L)(Y) = \coprod_{(Y,v)\in p^{-1}(Y)} L_{X\backslash\mathcal{C}}(Y,v) = \coprod_{v\,:\,X\to Y} L.$$

A $w : Y \to Y'$ defines a functor $p^{-1}(Y) \to p^{-1}(Y')$ by

$(Y,v) \mapsto (Y',wv)$, so $\mathcal{P}(X,L)$ is a system of coefficients on $\mathcal{C}$.

Now we want to compute $H_*(\mathcal{C},\mathcal{P}(X,L))$. We have

$$\begin{aligned}\mathcal{C}_k(\mathcal{C},\mathcal{P}(X,L)) &= \coprod_{X_0\to\ldots\to X_k} \Big(\coprod_{X\to X_0} L\Big) \\ &= \coprod_{X\to X_0\to\ldots\to X_k} L \\ &= \mathcal{C}_k(X\backslash\mathcal{C},L_{X\backslash\mathcal{C}}) .\end{aligned}$$

It follows that $H_k(\mathcal{C},\mathcal{P}(X,L)) = H_k(X\backslash\mathcal{C},L) = H_k(B(X\backslash\mathcal{C}),L)$. Now $X\backslash\mathcal{C}$ has initial object (X,id_X) so $B(X\backslash\mathcal{C})$ is contractible by (2.1) iv. Hence $\mathcal{P}(X,L)$ is acyclic.

We use the $\mathcal{P}(X,\mathcal{L}(X))$ to build an acyclic covering of $\mathcal{L}$. We have a morphism of functors

$$\mathcal{P}(X,\mathcal{L}(X)) \to \mathcal{L}$$

defined for $Y \in \mathcal{C}$ by

$$\mathcal{P}(X,\mathcal{L}(X))(Y) = \coprod_{v\,:\,X\to Y} \mathcal{L}(X) \xrightarrow{\Sigma\mathcal{L}(v)} \mathcal{L}(Y)$$

Now take

$$\mathcal{P} = \coprod_{X\in\mathcal{C}} \mathcal{P}(X,\mathcal{L}(X)) .$$

The induced morphism $\mathcal{P} \to \mathcal{L}$ is then surjective. ⊠

For a functor $f : \mathcal{C} \to \mathcal{C}'$ between small categories, Gabriel-Zisman [3] give in app.II theorem 3.6 a spectral sequence relating the homologies of $\mathcal{C}$ and $\mathcal{C}'$. We shall derive a similar spectral sequence, starting from an explicit description of the double complex. We use this double complex to compute the edge homomorphisms of the spectral sequence.

If $f : \mathcal{C} \to \mathcal{C}'$ is a functor, observe that $Y \mapsto H_q(f/Y,L)$ is a system of coefficients on $\mathcal{C}'$ for any abelian group L. In case the functor f is precofibred, it follows from the definition of cobase change that $Y \mapsto H_q(f^{-1}(Y),L)$ is a system of coefficients on $\mathcal{C}'$, and that it is equal to $Y \mapsto H_q(f/Y,L)$.

(3.3) Theorem. If $f : \mathcal{C} \to \mathcal{C}'$ is a functor between small categories, L an abelian group, we have a first quadrant spectral sequence

$$E^2_{pq} = H_p(\mathcal{C}',Y \mapsto H_q(f/Y,L)) \Rightarrow H_{p+q}(\mathcal{C},L).$$

Sketch of proof. We assume $L = \mathbb{Z}$. Let $\mathcal{C}_{**}(f)$ be the double complex such that $\mathcal{C}_{pq}(f)$ is the free abelian group on

$$\{(X_0 \to \ldots \to X_q, fX_q \to Y_0 \to \ldots \to Y_p) \mid X_i \in \mathcal{C},\ Y_i \in \mathcal{C}'\}.$$

The definition of the differentials is obvious.

The first spectral sequence of this double complex has as E^1-term

$$E^1_{pq}(I) = \coprod_{Y_0 \to \ldots \to Y_p} H_q(f/Y,\mathbb{Z}).$$

By the general theory of double complexes (cf. Cartan-Eilenberg [10] Ch XV §6) we know that $E^2_{pq}(I)$ is the homology of the chain complex $\mathcal{C}_*(\mathcal{C}',Y \to H_q(f/Y,\mathbb{Z})) = E^1_{*q}(I)$ and hence

$$E^2_{pq}(I) = H_p(\mathcal{C}',Y \mapsto H_q(f/Y,\mathbb{Z})).$$

The second spectral sequence has as E^1-term

$$E^1_{qp}(II) = \coprod_{X_0 \ldots X_q} H_p(fX_q\backslash\mathcal{C}',\mathbb{Z})$$

Because $B(fX_q\backslash\mathcal{C}')$ is contractible by (2.1)iv, we know that $H_p(fX_q\backslash\mathcal{C}',\mathbb{Z}) = 0$ for $p > 0$ and so

$$E^1_{*0}(II) = \mathcal{C}_*(\mathcal{C}^0,\mathbb{Z})$$

$$E^1_{*p}(II) = 0 \qquad p > 0$$

Hence we find

$$H_*(\mathcal{C}_{**}(f)) \cong H_*(\mathcal{C}^0,\mathbb{Z})$$
$$\cong H_*(\mathcal{C},\mathbb{Z})$$

To compute the edge homomorphisms of the first spectral sequence we observe that there is a morphism of complexes

$$\mathcal{C}_*((f/Y_0)^0) \to \mathcal{C}_{**}(f)$$

sending

$$(X_0,v_0) \leftarrow \dots \leftarrow (X_q,v_q) \mapsto (X_0 \to \dots \to X_q, fX_q \xrightarrow{v_q} Y_0).$$

It is clear that

$$H_q(f/Y_0,\mathbb{Z}) \to H_q(\mathcal{C}_{**}(f))$$

factorises through the edge homomorphism $E^2_{0q}(I) \to H_q(\mathcal{C}_{**}(f))$, i.e. the diagram below commutes

$$\begin{array}{ccc} H_q(f/Y_0,\mathbb{Z}) & \longrightarrow & H_q(\mathcal{C}_{**}(f)) \\ & \searrow \qquad \nearrow_{\text{edge}} & \\ & E^2_{0q}(I) = H_0(\mathcal{C}', Y \mapsto H_q(f/Y,\mathbb{Z})) & \end{array}$$

and careful inspection shows that

$$H_q((f/Y_0)^0,\mathbb{Z}) \to H_q(\mathcal{C}_{**}(f)) \to H_q(\mathcal{C}^0,\mathbb{Z})$$

is also the map induced by the functor $(f/Y_0)^0 \to \mathcal{C}^0$.

Now we want to compute the other edge, viz. $H_p(\mathcal{C}_{**}(f)) \to E^2_{p0}(I)$, under the assumption that $B(f/Y)$ is non-empty and connected for all $Y \in \mathrm{Obj}(\mathcal{C}')$. Note that this assumption implies that $E^2_{p0}(I) = H_p(\mathcal{C}',\mathbb{Z})$. We consider first the identity $id_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}$. Both spectral sequences of the double complex $\mathcal{C}_{**}(id_{\mathcal{C}})$ degenerate, and in this case the edge homomorphism

$$H_p(\mathcal{C}_{**}(id_{\mathcal{C}})) \to E^2_{p0}(I) = H_p(\mathcal{C},\mathbb{Z})$$

is also an isomorphism. Since the first spectral sequence is functorial, we have commutativity of

$$\begin{array}{ccc}
H_p(\mathcal{C}_{**}(id_{\mathcal{C}})) & \xrightarrow[\text{edge}]{\sim} & H_p(\mathcal{C},\mathbb{Z}) \\
\downarrow & & \downarrow H_p(f) \\
H_p(\mathcal{C}_{**}(f)) & \xrightarrow{\text{edge}} & H_p(\mathcal{C}',\mathbb{Z})
\end{array}$$

The left-hand vertical arrow is an isomorphism since $H_p(\mathcal{C}_{**}(id_{\mathcal{C}}))$ and $H_p(\mathcal{C}_{**}(f))$ are both isomorphic to $H_p(\mathcal{C}^{o},\mathbb{Z})$ by means of the second spectral sequence which is also functorial. Thus the edge $H_p(\mathcal{C}_{**}(f)) \to H_p(\mathcal{C}',\mathbb{Z})$ can be identified via this diagram with $H_p(f)$.

Observe that the first and second spectral sequences of $\mathcal{C}_{**}(id_{\mathcal{C}})$ yield an identification of $H_*(\mathcal{C},\mathbb{Z})$ and $H_*(\mathcal{C}^{o},\mathbb{Z})$. One can show that this is the usual identification, cf. (2.1)i. The reader can construct a proof himself from Quillen's proof of theorem A. We leave it as a (difficult) exercise, since we don't need it.

II. Partially ordered sets.

1. Definitions and generalities.

A partially ordered set (abbreviated poset) is a set X endowed with a reflexive, transitive relation $\leqslant$ such that $a \leqslant b$ and $b \leqslant a$ imply $a = b$. A morphism $f : X \to Y$ of posets is a map such that $f(x) \leqslant f(y)$ if $x \leqslant y$. A subposet $Y \subset X$ is a subset $Y \subset X$ with the inherited ordering.

We can view X as a small category taking X as a set of objects; a unique morphism $x \to y$ exists if and only if $x \leqslant y$. We can build its realisation as in I,§2. It is built of non-degenerate simplices, i.e. simplices coming from sequences $x_0 < \ldots < x_k$, $x_i \in X$, with appropriate identifications. We say it consists of simplices $x_0 < \ldots < x_k$.

In this section, we want to describe certain operations on posets, and investigate the relation between a poset X and a subposet Y in certain cases.

(1.1) <u>Definitions</u>. i) If X and Y are posets, the join $X * Y$ is defined as a set by $X \amalg Y$. The ordering on $X * Y$ extends the ordering on X and Y and for $x \in X$, $y \in Y$ we have $x \leqslant y$.

ii) For posets X,Y we define the poset $H(X,Y)$ as a set by $X \amalg Y \amalg X \times Y$. The ordering extends the ordering on X and Y and for $x,x' \in X$, $y,y' \in Y$ we have

$(x,y) \leqslant (x',y')$ in $H(X,Y)$ if and only if $x \leqslant x'$ in X,
$y \leqslant y'$ in Y.

$x \leqslant (x',y')$ in $H(X,Y)$ if and only if $x \leqslant x'$ in X.

$y \leqslant (x',y')$ in $H(X,Y)$ if and only if $y \leqslant y'$ in Y.

If X,Y are posets define $h(X,Y) : H(X,Y) \to X * Y$ for $x \in X$, $y \in Y$, and $(x,y) \in X \times Y$ by $h(X,Y)(x) = x$, $h(X,Y)(y) = y$, $h(X,Y)(x,y) = y$. From (1.1) we see $h(X,Y)$ is a morphism of posets. The following proposition gives some useful properties of $- * -$ and $H(-,-)$.

(1.2) Proposition. i) $X * Y$, $H(X,Y)$ are functorial, and $h(X,Y)$ is a morphism of functors.

ii) $|X * \{p\} * Y| = C|X * Y|$ for all posets X and Y.

iii) For all posets X and Y, $|h(X,Y)|$ is a homotopy equivalence.

iv) $|H(X,Y)| = \mathrm{Hopf}(|X|,|Y|)$ for all posets X and Y.

Proof. i) is obvious.

ii) The realisation of $X * \{p\} * Y$ consists of simplices $x_0 < x_1 < \ldots < x_k < y_0 < \ldots < y_l$, $x_i \in X$, $y_j \in Y$ with k,l possibly -1, though not both, and $x_0 < \ldots < x_k < p < y_0 < \ldots$ $\ldots < y_l$, $x_i \in X$, $y_j \in Y$ with k,l possibly -1. So the realisation of $|X * \{p\} * Y|$ is a cone over $|X * Y|$ with top p.

iv) $|H(X,Y)|$ consists of simplices of the following two forms:

$$x_0 < \ldots < x_k < (x_{k+1},y_{k+1}) < \ldots < (x_n,y_n),\ x_i \in X,\ y_j \in Y,\quad n \geq 0,\ -1 \leq k \leq n$$

and $$y_0 < \ldots < y_k < (x_{k+1},y_{k+1}) < \ldots < (x_n,y_n),\ x_i \in X,\ y_j \in Y,\quad n \geq 0,\ -1 \leq k \leq n.$$

From this it follows, if we take $X \cup X \times Y$ and $Y \cup X \times Y$ as subposets of the poset $H(X,Y)$ that

$$|H(X,Y)| = |X \cup X \times Y| \cup_{|X\times Y|} |Y \cup X \times Y| .$$

Now the map $X \times (\{p\} * Y) \to X \cup X \times Y$ sending $(x,p) \mapsto x$, and $(x,y) \mapsto (x,y)$ is seen to be an isomorphism of posets so

$$|X \cup X \times Y| = |X \times (\{p\} * Y)|$$

$$= |X| \times |\{p\} * Y| \qquad \text{by I(2.1)(i)}$$
$$= |X| \times C|Y| \qquad \text{by (ii) above.}$$

Analogously

$$|Y \cup X \times Y| = C|X| \times |Y|.$$

So

$$|H(X,Y)| = |X| \times C|Y| \cup_{|X|\times|Y|} C|X| \times |Y|$$
$$= \mathrm{Hopf}(|X|,|Y|) \qquad \text{by definition}$$

This finishes the proof of (iv).

iii) By I, theorem (2.2) it is enough to show $|h(X,Y)/x|$ and $|h(X,Y)/y|$ are contractible for $x \in X$, $y \in Y$. Now

$$h(X,Y)/x = \{x' \in X | x' \leqslant x\}$$
$$= X/x$$

As x is a final object of X/x, we find by I (2.1)(iv) that $|X/x|$ is contractible. Also

$$h(X,Y)/y = X \cup (Y/y) \cup X \times (Y/y)$$

So by iv) above

$$|h(X,Y)/y| = \mathrm{Hopf}(|X|,|Y/y|)$$

which is contractible as $|Y/y|$ is contractible. ⊠

We want to investigate now the relation between $|X|$ and $|Y|$ for a subposet Y of a poset X, in certain special cases. We call a poset X discrete if $x \leqslant x'$ implies $x = x'$ for all $x,x' \in X$. If Y is a subposet of X, $x \in X$, we define

$$\mathrm{Link}_Y(x) = \{y \in Y | y < x \text{ or } y > x\}.$$
$$\mathrm{Link}_Y^+(x) = \{y \in Y | y > x\}$$
$$\mathrm{Link}_Y^-(x) = \{y \in Y | y < x\}.$$

Then obviously

$$\mathrm{Link}_Y(x) = \mathrm{Link}_Y^-(x) * \mathrm{Link}_Y^+(x).$$

If Y is a subposet of X, then $X\setminus Y$ is also a subposet. We require $X\setminus Y$ to be discrete. In this case we have

(1.3) Proposition. Let Y be a subposet of the poset X. Assume $X\setminus Y$ is discrete. Then

$$|X|/|Y| = \bigvee_{x\in X\setminus Y} S|\mathrm{Link}_Y(x)|$$

$$= \bigvee_{x\in X\setminus Y} S\ \mathrm{Hopf}(|\mathrm{Link}_Y^-(x)|,|\mathrm{Link}_Y^+(x)|)$$

Proof. Since $X\setminus Y$ is discrete, a simplex $x_0 < \ldots < x_n$ in the realisation of X contains at most one element in $X\setminus Y$, i.e. either for all i, $x_i \in Y$ or if $x_j \notin Y$ then for $i \neq j$, $x_i \in Y$. Hence, denoting $Y(x) = Y \cup \{x\}$ for $x \in X\setminus Y$ we find

$$|X| = \bigcup_{x\in X\setminus Y} |Y(x)|$$

and if $x,x' \in X\setminus Y$ we have $|Y(x)| \cap |Y(x')| = |Y|$. Contracting $|Y|$ to a point we find

$$|X|/|Y| = \bigvee_{x\in X\setminus Y} |Y(x)|/|Y|.$$

To compute $|Y(x)|/|Y|$, observethat, by the description of simplices in $|Y(x)|$ above

$$|Y(x)| = |Y| \cup_{|\mathrm{Link}_Y^-(x)*\mathrm{Link}_Y^+(x)|} |\mathrm{Link}_Y^-(x)*\{x\}*\mathrm{Link}_Y^+(x)|$$

$$= |Y| \cup_{|\mathrm{Link}_Y^-(x)*\mathrm{Link}_Y^+(x)|} C|\mathrm{Link}_Y^-(x)*\mathrm{Link}_Y^+(x)|$$

So we find

$$|Y(x)|/|Y| = S|\mathrm{Link}_Y^-(x) * \mathrm{Link}_Y^+(x)|$$

$$= S\ \mathrm{Hopf}(|\mathrm{Link}_Y^-(x)|,|\mathrm{Link}_Y^+(x)|) \text{ by (1.2)iii),iv).}$$

This finishes our proof. ⊠

2. Homogeneous posets and morphisms.

This section is devoted to a special class of posets and morphisms. Our aim is to describe these so-called homogeneous morphisms in detail. First some technicalities.

If X is a poset, a chain of length k in X is a sequence $x_0 < \ldots < x_k$ of elements in X. Chains of length k correspond obviously to k-simplices in the realisation. If the length of chains in X is bounded we call X finite dimensional. This holds if and only if $|X|$ is a finite dimensional complex. The dimension of $|X|$, denoted $\dim|X|$, is then equal to the maximal chain length in X. This last number is denoted by dX.Convention: if $X = \phi$, then $dX = -1$. If X is a finite dimensional poset, so is each subposet.

Let X be a finite dimensional poset. For $x \in X$ we define the heighth of x, denoted $ht_X(x)$ or $ht(x)$ if no confusion arises, by

$$ht_X(x) = 1 + d(\mathrm{Link}_X^-(x))$$

i.e. $ht_X(x)$ is the maximal length of a chain ending in x. Now define

(2.1) Definition. i) A poset X is called homogeneous of dimension d if X is finite dimensional, $dX = d$, and each chain $y_0 < \ldots < y_k$ in X can be refined to a chain of maximal length, i.e. there is a chain $x_0 < \ldots < x_d$ in X and a $\phi : \{0,\ldots,k\} \to \{0,\ldots,d\}$ with $\phi(i) < \phi(j)$ if $i < j$ such that $x_{\phi(i)} = y_i$.

ii) Let X,Y be posets, homogeneous of dimension d and let $f : X \to Y$ be a morphism. We call f homogeneous if $ht_Y f(x) = ht_X(x)$ for all $x \in X$ and f/y is a homogeneous poset of dimension $ht_Y(y)$ for all $y \in Y$.

iii) If moreover f is a bijection on the elements of heighth $\neq k$, then f is called homogeneous of degree k.

A homogeneous morphism has the lifting property for chains:

(2.2) Lemma. If $f : X \to Y$ is a homogeneous morphism between homogeneous posets of dimension d, then for each chain $y_0 < \dots < y_k$ in Y there is a chain $x_0 < \dots < x_k$ in X such that $f(x_i) = y_i$. In particular, f is surjective.

Proof. Induction on k. Suppose we have lifted $y_0 < \dots < y_{k-1}$ to a chain $x_0 < \dots < x_{k-1}$. The latter chain can be refined to a maximal chain in f/y_k so we can certainly find an $x_k \in f/y_k$ with $ht_{f/y_k}(x_k) = ht_Y(y_k)$ $(= ht_X(x_k))$ and such that $x_0 < \dots < x_k$ is a chain in X. Now $f(x_k) \leq y_k$ and $ht_Y f(x_k) = ht_X(x_k) = ht_Y(y_k)$ so $f(x_k) = y_k$. ⊠

We first describe homogeneous morphisms of degree k, and then we shall find a factorisation of a homogeneous morphism into homogeneous morphisms of certain degree.

Notation: if X is a finite dimensional poset, we define

$$X_{\leq k} = \{x \in X \mid ht(x) \leq k\}$$
$$X_{\geq k} = \{x \in X \mid ht(x) \geq k\}.$$

(2.3) Proposition. Let X,Y be homogeneous posets of dimension d, and $f : X \to Y$ homogeneous of degree k. Then there is a factorisation of f, say $X \xrightarrow{i} Z \xrightarrow{h} Y$ such that i makes X a sub-poset of Z, the map $|h|$ is a homotopy equivalence, and

$$|Z|/|X| = \bigvee_{\substack{y \in Y \\ ht(y)=k}} S\,\mathrm{Hopf}(|f/y|, |\mathrm{Link}_Y^+(y)|)$$

Proof. Define $Z = X_{\leq k} \amalg Y_{\geq k}$ as a set. Take on $X_{\leq k}$ and $Y_{\geq k}$ the induced ordering, and define for $x \in X_{\leq k}$, $y \in Y_{\geq k}$

$x < y$ in Z if and only if $f(x) \leqslant y$ in Y.

It is clear that this defines an ordering on Z. Define $i : X \to Z$ by $i(x) = x$ if $ht_X(x) \leqslant k$, $i(x) = f(x)$ if $ht_X(x) > k$.

To show i makes X a subposet of Z, we have to show that $i(x) < i(x')$ implies $x < x'$ for $x,x' \in X$. If $ht_X(x') \leqslant k$ there is nothing to prove. If $ht_X(x') > k$ we have $i(x') = f(x')$, and the definition of the ordering on Z implies $f(x) < f(x')$ both if $ht_X(x) \leqslant k$ and $ht_X(x) > k$. As in the proof of (2.2) we find an x'' with $x < x''$ a lifting of $f(x) < f(x')$. As $ht(f(x'')) = ht(f(x')) \neq k$ and $f(x') = f(x'')$ it follows that $x' = x''$ and $x < x'$. Now identify X and $i(X)$.

To prove the last assertion, observe that

$$Z \backslash X = \{y \in Y \mid ht_Y(y) = k\}$$

and if $ht_Y(y) = k$, then

$$\begin{aligned} \mathrm{Link}_X^-(y) &= \{x \in X \mid f(x) \leqslant y\} \\ &= f/y \end{aligned}$$

and as f is bijective on elements of heighth $\neq k$,

$$\begin{aligned} \mathrm{Link}_X^+(y) &= \{x \in X \mid y < f(x)\} \\ &= \{y' \in Y \mid y < y'\} \\ &= \mathrm{Link}_Y^+(y) \end{aligned}$$

because $\mathrm{Link}_X^+(y) \subset X_{\geqslant k+1}$. Now applying proposition (1.3) this yields

$$|Z|/|X| = \bigvee_{\substack{y \in Y \\ ht(y)=k}} S\,\mathrm{Hopf}(|f/y|, |\mathrm{Link}_Y^+(y)|)$$

It remains to define h and to show it is a homotopy equivalence. Of course, we define $h(x) = f(x)$ if $x \in X_{\leqslant k}$, $h(y) = y$ if $y \in Y_{\geqslant k}$. Then h is a morphism of posets. Also define $j : Y \to Z$ by $j(y) = y$ if $ht(y) \geq k$, and $j|Y_{\leq k-1}$ is the inverse of $f|X_{\leq k-1}$. By the lifting of chains, j is a morphism of posets. Moreover $hj = id_Y$ and $jh(z) \geq z$ for all $z \in Z$. So by I (2.1)(ii) the map $|h|$ is a homotopy equivalence. ☒

Now let $f : X \to Y$ be a homogeneous morphism between homogeneous posets X,Y of dimension d. We define the posets U_i for $i = 0,\dots,d+1$ as follows:

$$U_i = X_{\leq i-1} \amalg Y_{\geq i}$$

on $X_{\leq i-1}$ and $Y_{\geq i}$ we take the induced ordering, and for $x \in X_{\leq i-1}$ and $y \in Y_{\geq i}$ we define:

$$x < y \text{ in } U_i \text{ if and only if } f(x) < y \text{ in } Y.$$

It is evident U_i is a poset. To prove U_i is homogeneous of dimension d, we have to show that a chain in U_i can be refined to a maximal chain. So let $x_0 < \dots < x_k < y_0 < \dots < y_l$ be a chain in U_i. We have three cases: i) $k \geq 0$, $l = -1$, ii) $k = -1$, $l \geq 0$, iii) $k,l \geq 0$.

Case i) Refine $x_0 < \dots < x_k$ to a maximal chain in X and project the elements of height $\geq i$ to Y.

ii) Refine $y_0 < \dots < y_l$ to a maximal chain in Y, say $\tilde{y}_0 < \dots < \tilde{y}_d$. By (2.2) we can lift this chain to X, say to $x_0 < \dots < x_d$ and take as refinement $x_0 < \dots < x_{i-1} < \tilde{y}_i < \dots < \tilde{y}_d$

iii) Refine $f(x_0) < f(x_1) < \dots < f(x_k) < y_0 < \dots < y_l$ to a maximal chain $\tilde{y}_0 < \dots < \tilde{y}_d$ in Y. Then

$$x_0 < \dots < x_k < \tilde{y}_i < \dots < \tilde{y}_d$$

is a chain in U_i. Refine the chain $x_0 < \dots < x_k$ in $f/\tilde{y}_i$ to a maximal chain $\tilde{x}_0 < \dots < \tilde{x}_i$, which is possible since $f/\tilde{y}_i$ is homogeneous by definition. Take as the desired refinement

$$\tilde{x}_0 < \ldots < \tilde{x}_{i-1} < \tilde{y}_i < \ldots < \tilde{y}_d .$$

This proves U_i is homogeneous of dimension d, and we see $ht_X(x) = ht_{U_i}(x)$ if $x \in X_{\leqslant i-1}$, $ht_Y(y) = ht_{U_i}(y)$ if $y \in Y_{\geqslant i}$.

Now define $f_i : U_i \to U_{i-1}$ by

$$f_i(x) = x \quad \text{if } x \in X_{\leqslant i-2}$$

$$f_i(x) = f(x) \text{ if } x \in X_{\leqslant i-1} \setminus X_{\leqslant i-2}$$

$$f_i(y) = y \quad \text{if } y \in Y_{\geqslant i}$$

Then we have

(2.4) Theorem. Let X,Y be homogeneous posets of dimension d, $f : X \to Y$ homogeneous, then we have a factorisation

$$X = U_{d+1} \xrightarrow{f_{d+1}} U_d \to \ldots \to U_1 \xrightarrow{f_1} U_0 = Y$$

with f_i homogeneous of degree i-1, and if $y \in U_{i-1}$ has height i-1 we have

$$f_i/y = f/y$$

$$\mathrm{Link}^+_{U_{i-1}}(y) = \mathrm{Link}^+_Y(y) .$$

Proof. The last assertions are easy to verify. It remains to be shown f_i is homogeneous of degree i-1. For this, it suffices to compute f_i/u for all $u \in U_{i-1}$ and show it is homogeneous of dimension $ht_{U_{i-1}}(u)$. Well, if $x \in X_{\leqslant i-2}$, we find f_i/x is homogeneous since

$$f_i/x = X/x$$

and if $y \in Y_{\geqslant i-1}$

$$f_i/y = \{x \in X_{\leqslant i-1} | f(x) \leqslant y\} \cup \{y' \in Y_{\geqslant i} | y' \leqslant y\}$$

$$= (f/y)_{\leqslant i-1} \cup (Y/y)_{\geqslant i}$$

Now, one verifies that $f|f/y : f/y \to Y/y$ is a homogeneous morphism. Therefore the proof of the homogeneity of U_i applies and we find that f_i/y is homogeneous of dimension $ht_Y(y) = ht_{U_{i-1}}(y)$.

 ⊠

3. Posets and homology.

In this section, we present some tools to show certain homology groups of posets vanish. They are the homological interpretation of the results obtained so far in this chapter.

(3.1) Definition. A poset X is called n-spherical if its realisation is a homology-wedge of n-spheres over $\mathbb{Z}$.

By the universal coefficient theorem, if X is an n-spherical poset, then $|X|$ is a homology-wedge of n-spheres over A for each commutative ring A. In the sequel, homology with unspecified coefficients will always be homology over $\mathbb{Z}$. We have $H_i(X) = H_i(|X|)$ by I, §3.

(3.2) Theorem. Let X be a poset, $Y \subset X$ a subposet. Assume $X \setminus Y$ is discrete, and Y is n-spherical. Then

i) If for each $x \in X \setminus Y$, the poset $\mathrm{Link}_Y(X)$ is n-1-spherical then X is n-spherical.

ii) If for each $x \in X \setminus Y$, the poset $\mathrm{Link}_Y(X)$ is n-spherical, and

$$\bigoplus_{x \in X \setminus Y} \tilde{H}_n(\mathrm{Link}_Y(X)) \to \tilde{H}_n(Y)$$

is surjective, then X is n+1-spherical.

Proof. By proposition (1.3) we have

$$|X|/|Y| = \bigvee_{x \in X \setminus Y} S|\mathrm{Link}_Y(x)|.$$

As $|X|/|Y| \cong |X| \cup_{|Y|} C|Y|$ we find a Mayer-Vietoris sequence

$$\dots \to \tilde{H}_i(|Y|) \to \tilde{H}_i(|X|) \to \bigoplus_{x \in X \setminus Y} \tilde{H}_i(S|\mathrm{Link}_Y(x)|) \to \tilde{H}_{i-1}(|Y|) \to \dots$$

Now we have $\tilde{H}_i(S|\mathrm{Link}_Y(x)|) = \tilde{H}_{i-1}(|\mathrm{Link}_Y(x)|)$ so:

$$\dots \to \tilde{H}_i(|Y|) \to \tilde{H}_i(|X|) \to \bigoplus_{x \in X \setminus Y} \tilde{H}_{i-1}(|\mathrm{Link}_Y(x)|) \to \tilde{H}_{i-1}(|Y|) \to \dots$$

This yields the desired result both for i) and ii), since one can prove that the connecting homomorphism

$H_i(S|\mathrm{Link}_Y(x)|) \to H_{i-1}(|Y|)$ is equal to the mapping

$H_{i-1}(|\mathrm{Link}_Y(x)|) \to H_{i-1}(|Y|)$. ⊠

Our next theorem gives a homological translation of (2.4).

(3.3) Theorem. Let $f : X \to Y$ be a homogeneous morphism between homogeneous posets of dimension d. Suppose Y is d-spherical, and for all $y \in Y$ that f/y is ht(y)-spherical, $\mathrm{Link}_Y^+(y)$ is (d-ht(y)-1)-sperical. Then X is d-spherical and there is a filtration

$$0 = F_{d+1} \subseteq F_d \subseteq \dots \subseteq F_0 \subseteq F_{-1} = \tilde{H}_d(X)$$

such that

$$F_{-1}/F_0 \cong \tilde{H}_d(Y) \quad \text{naturally,}$$

$$F_q/F_{q+1} \cong \bigoplus_{\substack{y\in Y\\ ht(y)=q}} \tilde{H}_q(f/y) \otimes \tilde{H}_{d-q-1}(\mathrm{Link}_Y^+(y)) \text{ for } q = 0,\dots,d.$$

Proof. Theorem (2.4) yields a factorisation

$$X = U_{d+1} \xrightarrow{f_{d+1}} U_d \to \dots \to U_1 \xrightarrow{f_1} U_0 = Y$$

with $f_k : U_k \to U_{k-1}$ homogeneous of degree k-1. Applying the exact sequence in the proof of (3.2) to the factorisation of f_k given in (2.3) we find

$$\dots \to \tilde{H}_i(|U_{k-1}|) \to \bigoplus_{\substack{y\in Y\\ ht(y)=k-1}} \tilde{H}_{i-1}(\mathrm{Hopf}(|f/y|,|\mathrm{Link}_Y^+(y)|)) \to$$

$$\to \tilde{H}_{i-1}(|U_k|) \to \tilde{H}_{i-1}(|U_{k-1}|) \to \dots$$

Now if ht(y) = k-1 we have f/y is k-1-spherical and $\mathrm{Link}_Y^+(y)$ is d-k-spherical, so $\mathrm{Hopf}(|f/y|,|\mathrm{Link}_Y^+(y)|)$ is a homology-wedge of d-spheres by I (1.5). Inductively we find U_k to be d-spherical, so X is, and, again by I (1.5). we have the exact sequence

$$0 \to \bigoplus_{\substack{y\in Y \\ ht(y)=k-1}} \tilde{H}_{k-1}(f/y) \otimes \tilde{H}_{d-k}(\mathrm{Link}_Y^+(y)) \to \tilde{H}_d(U_k) \to \tilde{H}_d(U_{k-1}) \to 0$$

So by induction we construct filtrations on the $\tilde{H}_d(U_k)$ yielding the desired filtration on $\tilde{H}_d(U_{d+1}) = \tilde{H}_d(X)$. ⊠

The hypotheses of theorem (3.3) are related to the notion of Cohen - Macaulay posets. We present here a definition suited to our purposes.

(3.4) Definition. A poset X is called Cohen-Macaulay of dimension d (abbreviated d-dim CM) if X is d-dimensional, d-spherical, and if for all $x,x' \in X$ such that $x < x'$ the following holds:

$\mathrm{Link}_X^-(x)$ is $(ht(x)-1)$-spherical

$\mathrm{Link}_X^+(x)$ is $(d-ht(x)-1)$-spherical

$(x,x') = \mathrm{Link}_X^+(x) \cap \mathrm{Link}_X^-(x')$ is $(ht(x)-ht(x')-2)$-spherical

In particular, a d-dim CM poset is homogeneous of dimension d. Observe that Quillen [11] §8 uses a different notion of CM-ness.

4. A special class of posets.

First some generalities about semi-simplicial sets. Let A_* be a semi-simplicial set. A simplex $a \in A_n$ is called degenerate if there exists $b \in A_{n-1}$ and a j such that $\sigma_j(b) = a$. If not, $a \in A_n$ is called non-degenerate. Take $A'_n \subseteq A_n$ the set of non-degenerate simplices, and define

$$\mathcal{O}(A_*) = \bigcup_{n=0}^{\infty} A'_n$$

Order $\mathcal{O}(A_*)$ as follows: if $a \in A'_n$, $b \in A'_{n+t}$, then $a \leqslant b$ if and only if there is a sequence $i_1,\dots,i_t$ such that $\partial_{i_1}\cdot\dots\cdot\partial_{i_t}(b)=a$.

We call $\mathcal{O}(A_*)$ the associated poset of A_*. The realisation $|A_*|$ of A_* (cf. I §2 or Segal [12]) has the following property.

(4.1) Theorem. If A_* is a semi-simplicial set, then $|A_*|$ and $|\mathcal{O}(A_*)|$ are homeomorphic.

This is just barycentric subdivision. Intuitively this is clear. For a detailed discussion, see Spanier [13] Ch.3 sec.3. Observe that $H_*(|A_*|) = H_*(\mathcal{O}(A_*))$.

We shall now define a class of semi-simplicial sets which at first glance looks quaint, and find the associated poset as above. However, once we come to examples, our construction will become more natural.

If V is a set, one can make V into a category by taking exactly one morphism between each pair of elements of V. The associated semi-simplicial set N_*V looks like

$$N_kV = \{(v_0,\dots,v_k) | v_i \in V\}$$

with

$$\partial_i(v_0,\dots,v_i,\dots,v_k) = (v_0,\dots,\hat{v}_i,\dots,v_k) \quad i = 0,\dots,k$$

$$\sigma_i(v_0,\dots,v_i,\dots,v_k) = (v_0,\dots,v_i,v_i,\dots,v_k) \quad i = 0,\dots,k.$$

We take $\tilde{N}_kV$ to be the subset of N_kV consisting of the sequences $(v_0,\dots,v_k)$ such that if $v_i = v_l$ for $i < l$ this implies $v_i = v_j = v_l$ for $i \leq j \leq l$. As $\tilde{N}_*V$ is closed under boundaries and degeneracies, $\tilde{N}_*V$ is a semi-simplicial set. Define

$$\mathcal{O}(V) = \mathcal{O}(\tilde{N}_*(V)) .$$

An element $(v_0,\dots,v_k) \in \tilde{N}_k(V)$ is degenerate if there is a k such that $v_k = v_{k+1}$, or, equivalently, by the definition of $\tilde{N}_k(V)$ if there are i,l such that $i \neq l$ and $v_i = v_l$. So

$$\mathcal{O}(V) = \{(v_0,\dots,v_k) | v_i \in V \text{ and } v_i \neq v_j \text{ if } i \neq j\}.$$

It is easily seen that the ordering on $\mathcal{O}(V)$ is as follows:

$$(v_0,\dots,v_k) \leqslant (w_o,\dots,w_l)$$

if and only if there is a $\phi : \{0,\dots,k\} \to \{0,\dots,l\}$ satisfying $\phi(i) < \phi(j)$ if $i < j$, such that $v_i = w_{\phi(i)}$ for $i = 0,\dots,k$. We abbreviate sometimes $\vec{v} = (v_0,\dots,v_k)$.

We wish to consider sub-semi-simplicial sets F_* of $\tilde{N}_*V$. Now $(v_0,\dots,v_p) \in F_p$ is degenerate in F_* if and only if it is degenerate in $\tilde{N}_*V$. As a result

$$\begin{aligned}\mathcal{O}(F_*) &= \{(v_0,\dots,v_p) \in \mathcal{O}(V) \mid (v_0,\dots,v_p) \in F_p,\ p \geqslant 0\}\\ &= \mathcal{O}(V) \cap F_*\end{aligned}$$

Because $\mathcal{O}(F_*)$ is closed under boundary operators, we have if $\vec{v} \in \mathcal{O}(F_*)$ and $\vec{w} \leqslant \vec{v}$ then $\vec{w} \in \mathcal{O}(F_*)$. We want to have a name for this property.

(4.2) <u>Definition</u>. Let V be a set, $F \subseteq \mathcal{O}(V)$ be a subposet. If $\vec{v} \in F$, $\vec{w} \leqslant \vec{v}$ implies $\vec{w} \in F$, we say F has the chain property or satisfies the chain condition.

Take $F \subseteq \mathcal{O}(V)$ having the chain property. We want to associate to F a sub-semi-simplicial set of $\tilde{N}_*V$. Intuitively it must consist of sequences made from elements of F by repeating each item often enough, to make it closed under degeneracies. More precisely:

$$N_k^!(F) = \{\vec{v} \in \tilde{N}_k(V) \mid \text{there is } \vec{w} \in F,\ i_1,\dots,i_t, \text{ such that } \sigma_{i_1},\dots,\sigma_{i_t}(\vec{w}) = \vec{v}\}$$

It is a technical exercise to show $N_*^!(F)$ is a semi-simplicial set and that the following holds.

(4.3) <u>Proposition</u>. Let V be a set. Then $\mathcal{O}$ and $N_*^!$ give a one-one correspondence between sub-semi-simplicial sets F_* of $\tilde{N}_*V$ and subposets F of $\mathcal{O}(V)$ having the chain property. We have

$$\mathcal{O}(N_*^!(F)) = F$$

$$N_*^!(\mathcal{O}(F_*)) = F_*$$

From now on, when $F \subseteq \mathcal{O}(V)$ has the chain property, we write F_* for the corresponding semi-simplicial set $N_*^!(F)$.

For $F \subseteq \mathcal{O}(V)$ satisfying the chain condition we have

$$F_{\leqslant p} = \{(v_0,\ldots,v_k) \in F \mid k \leqslant p\}.$$

We find $(F_{\leqslant p})_*$ to be a semi-simplicial set whose realisation consists of the i-simplices in $|F_*|$ with $i \leqslant p$, i.e. $|(F_{\leqslant p})_*| = |F_*|_p$, the p-skeleton of $|F_*|$.

For $(v_0,\ldots,v_p) \in F$ we have

$$ht_F(v_0,\ldots,v_p) = p.$$

So if we suppose in addition F is homogeneous of dimension d for a certain d, then for $(v_0,\ldots,v_p) \in F$ there is $(w_0,\ldots,w_d) \in F$ with $(v_0,\ldots,v_p) \leqslant (w_0,\ldots,w_d)$. In this case, $|F_*|$ is a d-dimensional cellular complex.

(4.4) Proposition. Let $F \subseteq \mathcal{O}(V)$ satisfy the chain condition. Suppose F is homogeneous of dimension d.

i) If F is d-spherical, then $F_{\leqslant p}$ is p-spherical for $p \leqslant d$.

ii) If $F_{\leqslant d-1}$ is d-1-spherical, and

$$\operatorname{im}(\tilde{H}_{d-1}(F_{\leqslant d-1}) \to \tilde{H}_{d-1}(F)) = 0$$

then F is d-spherical.

Proof. i) By (4.1),(4.3) we have $|F| = |F_*|$, $|F_{\leqslant p}| = |(F_{\leqslant p})_*|$. Above we saw that $|F_*|_p = |(F_{\leqslant p})_*|$. As the p-skeleton of a d-spherical cellular complex is p-spherical for $p \leqslant d$ we are through.

ii) Again $|F| = |F_*|$, $|F_{\leq d-1}| = |(F_{\leq d-1})_*|$, and $|(F_{\leq d-1})_*|$ is the d-1-skeleton of the d-dimensional complex $|F_*|$. So $\tilde{H}_{d-1}(|(F_{\leq d-1})_*|) = \tilde{H}_{d-1}(F_{\leq d-1})$ surjects onto $\tilde{H}_{d-1}(|F_*|) = \tilde{H}_{d-1}(F)$. It follows that $\tilde{H}_{d-1}(F) = 0$. As $|F|$ is d-dimensional, $\tilde{H}_d(F)$ is free over $\mathbb{Z}$. ⊠

We shall be concerned with subsets $F \subseteq \mathcal{O}(V)$ satisfying one extra condition.

(4.5) Definition. A subposet $F \subseteq \mathcal{O}(V)$ is called full of dimension d if the following holds:

i) F satisfies the chain condition.

ii) F is homogeneous of dimension d.

iii) If $(v_0,\ldots,v_p) \in F$, $\sigma \in S_{p+1}$, then $(v_{\sigma 0},\ldots,v_{\sigma p}) \in F$. This is called the symmetry condition.

The posets we consider in chapter III are of this type. We introduce some of them first.

Examples. 1) If $V = \{1,\ldots,n\}$ then $\mathcal{O}(V)$ is full of dimension n-1. We denote $\mathcal{O}(V) = \mathcal{O}(n)$.

2) Let R be a ring, always assumed to be commutative with 1. Take $\mathcal{O}(n,R)$ to consist of all sequences $(v_0,\ldots,v_p) \in \mathcal{O}(R^n)$ such that there is an extension of this sequence to an R-basis of R^n, or, alternatively, writing the v_i as column vectors, such that there exist $v_{p+1},\ldots,v_{n-1}$ with $(v_0,\ldots,v_p,v_{p+1},\ldots,v_{n-1}) \in GL_n(R)$. The poset $\mathcal{O}(n,R)$ is full of dimension n-1.

In particular, for a field K, $\mathcal{O}(n,K)$ consists of sequences of linearly independent vectors in K^n.

3) For a ring R, define

$$\mathcal{A}(n,R) = \{(v_0,\ldots,v_i) \in \mathcal{O}(R^n) \mid (v_1-v_0,\ldots,v_i-v_0) \in \mathcal{O}(n,R)\}$$

$\mathcal{A}(n,R)$ is full of dimension n.

We finish this section with a useful

(4.6) Lemma. Let $F \subseteq \mathcal{O}(V)$ satisfy the chain condition. If $(v_0,\ldots,v_p) \in F$ then

$$|\mathrm{Link}_F^-(v_0,\ldots,v_p)| = S^{p-1}$$

Proof. It is sufficient to consider the case $F = \mathcal{O}(p+1)$ and $(v_0,\ldots,v_p) = (1,\ldots,p+1)$. Then

$$\mathrm{Link}_F^-(v_0,\ldots,v_p) = (\mathcal{O}(p+1)/(1,\ldots,p+1))_{\leq p-1}$$

Define Δ_p to be the ordered set of positive integers $\leq p+1$. As is well known $|\Delta_p| = \Delta^p$, the standard p-simplex and $|\Delta_p|_{p-1} = S^{p-1}$. Now we have

$$\begin{aligned} N_k\Delta_p &= \{(i_0,\ldots,i_k) \mid 1 \leq i_j \leq p+1,\ i_j \leq i_{j+1}\} \\ &= N_k'(\mathcal{O}(p+1)/(1,\ldots,p+1)). \end{aligned}$$

So we find

$$N_*\Delta_p = \mathcal{O}(p+1)/(1,\ldots,p+1)_*$$

(compare (4.3)) and hence

$$\begin{aligned} |\mathrm{Link}_F^-(v_0,\ldots,v_p)| &= |((\mathcal{O}(p+1)/(1,\ldots,p+1))_{\leq p-1})_*| \\ &= |(\mathcal{O}(p+1)/(1,\ldots,p+1))_*|_{p-1} \\ &= |N_*\Delta_p|_{p-1} \\ &= S^{p-1} \end{aligned}$$

which ends our proof. ☒

5. The Z_n-construction.

In this section, F is a full subset of $\mathcal{O}(V)$ of dimension d. Take $(v_0,\ldots,v_p) \in F$ and define

$$F_{(v_0,\ldots,v_p)} = \{(w_0,\ldots,w_q) \mid (v_0,\ldots,v_p,w_0,\ldots,w_q) \in F\}.$$

Evidently $F_{(v_0,\dots,v_p)} \subseteq F$. From the definition (4.5), it follows easily $F_{(v_0,\dots,v_p)}$ is full of dimension d-p-1. We have a sort of "transitivity property"

(5.1) $(F_{(v_0,\dots,v_p)})_{(v_{p+1},\dots,v_q)} = F_{(v_0,\dots,v_q)}$
for $(v_0,\dots,v_q) \in F$. These subsets of F play a crucial role in our discussion.

Consider now $\mathrm{Link}^+_F(v_0,\dots,v_p)$. The symmetry condition implies that deleting $(v_0,\dots,v_p)$ gives rise to a morphism

$$\mathrm{Link}^+_F(v_0,\dots,v_p) \to F_{(v_0,\dots,v_p)}$$

On the other hand, we can construct $\mathrm{Link}^+_F(v_0,\dots,v_p)$ from $F_{(v_0,\dots,v_p)}$ by inserting $(v_0,\dots,v_p)$ in the correct order in the sequences of $F_{(v_0,\dots,v_p)}$. The Z_n-construction is just the abstract formulation of this process. We give a formal definition first and then proceed to investigate some of its properties.

Take n+1 elements $z_0,\dots,z_n$, all distinct and not in V. Let $V_n = V \cup \{z_0,\dots,z_n\}$. There is a projection

$$\zeta_n : \mathcal{O}(V_n)\backslash\mathcal{O}(\{z_0,\dots,z_n\}) \to \mathcal{O}(V)$$

given by deleting the z_i. Let $\vec{z} = (z_0,\dots,z_n)$, and define

$$Z_nF = \{\vec{v} \in \mathcal{O}(V_n)\backslash(\mathcal{O}(\{z_0,\dots,z_n\}) \mid \vec{z} \leqslant \vec{v},\ \zeta_n\vec{v} \in F\}.$$

So $Z_{-1}F = F$. The restriction of ζ_n to Z_nF defines a projection $Z_nF \to F$, again denoted ζ_n. We have a section $\psi_n : F \to Z_nF$, defined by

$$\psi_n : (v_0,\dots,v_p) \mapsto (v_0,\dots,v_p,z_0,\dots,z_n) .$$

Observe that indeed we have for $(v_0,\dots,v_p) \in F$

$$\mathrm{Link}^+_F(v_0,\dots,v_p) \cong Z_p(F_{(v_0,\dots,v_p)}).$$

The main result on "Z_n" is the description of ζ_n given in (5.2).

But first we have to make some preparations.

If $\vec{w} = (w_0,\ldots,w_m) \in Z_nF$ then since $\vec{z} \leqslant \vec{w}$ there is a $\phi : \{0,\ldots,n\} \to \{0,\ldots,m\}$ such that $w_{\phi(i)} = z_i$. We define the position pos_{z_0} of z_0 in $\vec{w}$ by $\mathrm{pos}_{z_0}(\vec{w}) = \phi(0)$. If there is an $i > \phi(0)$ such that $w_i \in V$ we call $\vec{w}$ of general type. If not we call $\vec{w}$ special. In fact $\vec{w}$ is special if and only if $\vec{w} \in \psi_n(F)$. Observe that for $(v_0,\ldots,v_p) \in F$ we have $\mathrm{pos}_{z_0}(\psi_n(v_0,\ldots,v_p)) = p+1$, and that Z_nF is the disjoint union of the set of elements of general type and ψ_nF. We want to use these facts to construct a filtration on Z_nF.

Define

$$X_0 = \{\vec{w} \in Z_nF \mid \vec{w} \text{ of general type}\}$$

and for $q = 1,\ldots,d+1$

$$X_q = X_0 \cup \psi_n(F_{\leqslant q-1})$$

Evidently $X_{d+1} = Z_nF$, so we have a filtration

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{d+1} = Z_nF$$

We can now state our theorem.

(5.2) <u>Theorem</u>. Let $F \subseteq \mathcal{O}(V)$ be full of dimension d, $n \geqslant 0$. Then we have a filtration as defined above.

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{d+1} = Z_nF$$

on Z_nF such that

$$|X_0| \cong |Z_{n-1}F|$$

$$|X_q|/|X_{q-1}| \cong \bigvee_{(v_0,\ldots,v_{q-1})\in F} S^q|Z_{n-1}F(v_0,\ldots,v_{q-1})|$$

for $q = 1,\ldots,d$ and

$$|X_{d+1}|/|X_d| \cong \bigvee_{(v_0,\ldots,v_d)\in F} S^d$$

Furthermore, $\zeta_n : Z_nF \to F$ admits a section.

Proof. The section ψ_n was constructed above.

Now consider X_0. By taking $(z_1,\ldots,z_n)$ we can form $Z_{n-1}F$. There is an injection

$$i : Z_{n-1}F \to X_0$$

given by sending $(w_0,\ldots,w_p) \in Z_{n-1}F$ to $(z_0,w_0,\ldots,w_p)$.

We wish to define a projection $\pi : X_0 \to Z_{n-1}F$ which is to give a homotopy inverse in the realisation. To define π, take $(w_0,\ldots,w_p) \in X_0$. Assume $\mathrm{pos}_{z_0}(w_0,\ldots,w_p) = j$. Then define

$$\pi(w_0,\ldots,w_p) = (w_{j+1},\ldots,w_p).$$

Because $(w_0,\ldots,w_p)$ is of general type, the sequence $(w_{j+1},\ldots,w_p)$ contains an element of V, so by the chain condition it lives in $Z_{n-1}F$. It is easily seen π is a morphism of posets such that $\pi i = \mathrm{id}_{Z_{n-1}F}$ and $i\pi(\vec{w}) \leqslant \vec{w}$ for $\vec{w} \in X_0$. Then I(2.1)(ii) shows $|i|$ and $|\pi|$ are homotopy inverses of each other. This shows $|X_0| \cong |Z_{n-1}F|$.

We want to use (1.3) to describe $|X_q|/|X_{q-1}|$ for $q = 1,\ldots,d+1$. So we first consider $X_q \backslash X_{q-1}$. As $X_i = X_0 \cup \psi_n(F_{\leqslant i-1})$, $X_q\backslash X_{q-1} = \psi_n(F_{\leqslant q-1}\backslash F_{\leqslant q-2})$ or

$$X_q\backslash X_{q-1} = \psi_n\{(v_0,\ldots,v_{q-1}) \in F\}$$

So $X_q\backslash X_{q-1}$ is discrete. To use (1.3) we have to compute "Links" first. We have

$$\mathrm{Link}^-_{X_{q-1}}\psi_n(v_0,\ldots,v_{q-1}) = \psi_n\mathrm{Link}^-_F(v_0,\ldots,v_{q-1})$$

and applying (4.6) we have

$$|\mathrm{Link}^-_{X_{q-1}}\psi_n(v_0,\ldots,v_{q-1})| = S^{q-2}.$$

On the other hand

$$\mathrm{Link}^+_{X_{q-1}}(\psi_n(v_0,\ldots,v_{q-1})) = \mathrm{Link}_{X_0}(\psi_n(v_0,\ldots,v_{q-1}))$$

since $X_{q-1} = X_0 \cup \psi_n(F_{\leqslant q-2})$. Just as we compared X_0 and $Z_{n-1}F$, we want to compare $\mathrm{Link}_{X_0}(\psi_n(v_0,\ldots,v_{q-1}))$ and $Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})$.

We think of $Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})$ as constructed with the elements $z_1,\ldots,z_n$. The map

$$i_{(v_0,\ldots,v_{q-1})} : Z_{n-1}(F_{(v_0,\ldots,v_{q-1})}) \to \mathrm{Link}_{X_0}(\psi_n(v_0,\ldots,v_{q-1}))$$

given by

$$i_{(v_0,\ldots,v_{q-1})}(w_0,\ldots,w_p) = (v_0,\ldots,v_{q-1},z_0,w_0,\ldots,w_p)$$

is a morphism of ordered sets. To define a homotopy inverse, take $(w_0,\ldots,w_p) \in \mathrm{Link}_{X_0}(\psi_n(v_0,\ldots,v_{q-1}))$ and define $\pi_{(v_0,\ldots,v_{q-1})}(w_0,\ldots,w_p) = \pi(w_0,\ldots,w_p)$. One readily sees $\pi_{(v_0,\ldots,v_{q-1})}(w_0,\ldots,w_p) \in Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})$. Furthermore $\pi_{(v_0,\ldots,v_{q-1})}\, i_{(v_0,\ldots,v_{q-1})} = \mathrm{id}_{Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})}$ and for all $\vec{w} \in \mathrm{Link}_{X_0}(\psi_n(v_0,\ldots,v_{q-1}))$ we have $i_{(v_0,\ldots,v_{q-1})}\pi_{(v_0,\ldots,v_{q-1})}(\vec{w}) \leqslant \vec{w}$. Again by I (2.1)(ii) we find $|i_{(v_0,\ldots,v_{q-1})}|$ and $|\pi_{(v_0,\ldots,v_{q-1})}|$ are homotopy inverses, so

$$|\mathrm{Link}^{+}_{X_{q-1}}(\psi_n(v_0,\ldots,v_{q-1}))| \cong |Z_{n-1}F_{(v_0,\ldots,v_{q-1})}|.$$

By (1.3) we now find for $q = 1,\ldots,d+1$

$$\begin{aligned}|X_q|/|X_{q-1}| &\cong \bigvee_{(v_0,\ldots,v_{q-1})\in F} S(\mathrm{Hopf}(|\mathrm{Link}^{-}_{X_{q-1}}(\psi_n(v_0,\ldots,v_{q-1}))|,\\ &\qquad\qquad |\mathrm{Link}^{+}_{X_{q-1}}(\psi_n(v_0,\ldots,v_{q-1}))|))\\ &\cong \bigvee_{(v_0,\ldots,v_{q-1})\in F} S\,\mathrm{Hopf}(S^{q-2},|Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})|)\\ &\cong \bigvee_{(v_0,\ldots,v_{q-1})\in F} S^q|Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})|\end{aligned}$$

In case $q = d+1$, $Z_{n-1}(F_{(v_0,\ldots,v_d)}) = \phi$ and $S^{d+1}(\phi) = S^d$. ⊠

In the following corollary, we give sufficient conditions for the sphericity of Z_nF.

(5.3) Corollary. Let $F \subseteq \mathcal{O}(V)$ be full of dimension d. Assume F is d-spherical, and for all $(v_0,\ldots,v_q) \in F$ that $F_{(v_0,\ldots,v_q)}$ is (d-q-1)-spherical. Then Z_nF is d-spherical and $\tilde{H}_d(\zeta_n) : \tilde{H}_d(Z_nF) \to \tilde{H}_dF$ is split surjective.

Proof. Induction on n. The case n = -1 being trivial, assume $n \geq 0$. Let

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_d \subseteq X_{d+1} = Z_nF$$

be the filtration on Z_nF of theorem (5.2).

By (5.2), $|X_0| \cong |Z_{n-1}F|$. As $Z_{n-1}F$ is d-spherical by the induction hypothesis, so is X_0. Hence it is enough to show that X_q is d-spherical if X_{q-1} is.

Assume X_{q-1} is d-spherical. By (5.2) X_q is also d-spherical if for all $(v_0,\ldots,v_{q-1}) \in F$ the poset $Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})$ is d-q-spherical.

For $(v_q,\ldots,v_r) \in F_{(v_0,\ldots,v_{q-1})}$ we have by (5.1)

$$(F_{(v_0,\ldots,v_{q-1})})_{(v_q,\ldots,v_r)} = F_{(v_0,\ldots,v_r)}$$

so $(F_{(v_0,\ldots,v_{q-1})})_{(v_q,\ldots,v_r)}$ is d-r-1 = (d-q)-(r-q+1)-spherical.

We now apply the induction hypothesis to $Z_{n-1}(F_{(v_0,\ldots,v_{q-1})})$ to show it is d-q-spherical.

Finally, the section ψ_n of ζ_n yields the desired splitting. ⊠

We finish this chapter with a sketchy account of the relation between CM and the Z_n-construction.

(5.4) Corollary. Let $F \subseteq \mathcal{O}(V)$ be full of dimension d. Then F is d-dim CM if and only if F is d-spherical and for all $(v_0,\ldots,v_q) \in F$, the poset $F_{(v_0,\ldots,v_q)}$ is d-q-1-spherical.

Proof. "only if". Since F is CM, $\mathrm{Link}_F^{+}(v_0,\ldots,v_q) = Z_qF_{(v_0,\ldots,v_q)}$

is d-q-1-spherical. Since the projection

$$\zeta_n : Z_q F_{(v_0,\dots,v_q)} \to F_{(v_0,\dots,v_q)}$$

splits, $F_{(v_0,\dots,v_q)}$ is d-q-1-spherical too.

"<u>if</u>". We have to show F satisfies the requirements of (3.4). Well:

$$|\mathrm{Link}^-_F(v_0,\dots,v_q)| \cong S^{q-1}$$

by (4.6) and

$$\mathrm{Link}^+_F(v_0,\dots,v_q) \cong Z_q F_{(v_0,\dots,v_q)}$$

which is d-q-1-spherical by (5.3).

Remains to consider for $(v_0,\dots,v_p) < (w_0,\dots,w_q)$ the poset $((v_0,\dots,v_p),(w_0,\dots,w_q))$. We have

$$((v_0,\dots,v_p),(w_0,\dots,w_q)) \cong (\mathcal{O}(q-p)/(1,\dots,q-p))_{\leq q-p-2}$$
$$\cong \mathrm{Link}^-_{\mathcal{O}(q-p)}(1,\dots,q-p)$$

as posets. This is an exercise we leave to the reader. It follows by (4.6) that $((v_0,\dots,v_p),(w_0,\dots,w_q))$ is q-p-2-spherical, as required. ⊠

Combining (5.3) and (5.4) we finally find

(5.5) <u>Corollary</u>. If $F \subseteq \mathcal{O}(V)$, full of dimension d, is d-dim CM, then $Z_n F$ is d-spherical for all n.

6. The < >-construction.

Let F be a full subposet of $\mathcal{O}(V)$ of dimension d. If S is a non-empty set, define

$$F < S > = \{((v_0,s_0),\dots,(v_p,s_p)) \in \mathcal{O}(V\times S) \mid (v_0,\dots,v_p) \in F\}$$

One easily sees that F < S > is again full of dimension d. We have the following theorem.

(6.1) Theorem. Let F be a full subposet of $\mathcal{O}(V)$ of dimension d. If F is d-dim CM then $F < S >$ is d-spherical for any non-empty set S.

Proof. Choose an element $s \in S$, and define $\sigma : F \to F < S >$ by $\sigma(v_0,\ldots,v_p) = ((v_0,s),\ldots,(v_p,s))$. It is clear that σ is a morphism of posets.

Define

$$X_0 = \{((v_0,s_0),\ldots,(v_p,s_p)) \in F < S > | \exists_i s_i = s\}$$

and for $q = 1,\ldots,d+1$

$$X_q = X_0 \cup \{((v_0,s_0),\ldots,(v_p,s_p)) \in F < S > | \forall_i s_i \neq s, p < q\}$$

Then we state

(6.2) Claim a) X_0 is d-spherical.

b) For $q = 0,\ldots,d$, if we have that X_q is d-spherical, then X_{q+1} is also d-spherical.

As $X_{d+1} = F < S >$, proving the claim obviously settles the proof of (6.1).

Proof of (6.2) a) The map $\sigma : F \to F < S >$ maps F into X_0. On the other hand we have a projection

$$\pi : X_0 \to F$$

where $\pi((v_0,s_0),\ldots,(v_p,s_p))$ is the subsequence of $(v_0,\ldots,v_p)$ consisting of the v_i such that $s_i = s$. Then π is a morphism of posets such that $\pi\sigma = id_F$ and for $((v_0,s_0),\ldots,(v_p,s_p)) \in X_0$ we have that $\sigma\pi((v_0,s_0),\ldots,(v_p,s_p)) \leqslant ((v_0,s_0),\ldots,(v_p,s_p))$. By I (2.1)(ii) we find that $|\pi|$ and $|\sigma|$ are homotopy inverse to each other. As F is d-spherical, X_0 is d-spherical, i.e. a).

b) We have

$$X_{q+1}\backslash X_q = \{((v_0,s_0),\dots,(v_q,s_q)) \in F < S > |\forall_i s_i \neq s\}$$

hence $X_{q+1}\backslash X_q$ is discrete. We find by (3.2)(i) that X_{q+1} is d-spherical if X_q is d-spherical and for each $((v_0,s_0),\dots,(v_q,s_q)) \in X_{q+1}\backslash X_q$ the poset $\mathrm{Link}_{X_q}((v_0,s_0),\dots,(v_q,s_q))$ is d-1-spherical.

To prove this, observe

$$\mathrm{Link}^-_{X_q}((v_0,s_0),\dots,(v_q,s_q)) = \mathrm{Link}^-_{F<S>}((v_0,s_0),\dots,(v_q,s_q))$$

hence by (4.6)

$$|\mathrm{Link}^-_{X_q}((v_0,s_0),\dots,(v_q,s_q))| = S^{q-1}$$

So according to (1.3) and I (1.3) we have

$$\begin{aligned}
&|\mathrm{Link}_{X_q}((v_0,s_0),\dots,(v_q,s_q))| = \\
&= \mathrm{Hopf}(|\mathrm{Link}^-_{X_q}((v_0,s_0),\dots,(v_q,s_q))|,|\mathrm{Link}^+_{X_q}((v_0,s_0),\dots \\
&\qquad \dots,(v_q,s_q))|) \\
&= \mathrm{Hopf}(S^{q-1},|\mathrm{Link}^+_{X_q}((v_0,s_0),\dots,(v_q,s_q))|) \\
&\cong S^q|\mathrm{Link}^+_{X_q}((v_0,s_0),\dots,(v_q,s_q))|
\end{aligned}$$

So we are finished if we prove that $\mathrm{Link}^+_{X_q}((v_0,s_0),\dots,(v_q,s_q))$ is d-q-1-spherical. In case, $q = d$, we see $\mathrm{Link}^+_{X_q}((v_0,s_0),\dots,(v_q,s_q)) = \phi$, so -1-spherical. Hence assume $q < d$.

We have

$$\mathrm{Link}^+_{X_q}((v_0,s_0),\dots,(v_q,s_q)) = \mathrm{Link}_{X_0}((v_0,s_0),\dots,(v_q,s_q)).$$

We want to compare $\mathrm{Link}_{X_0}((v_0,s_0),\dots,(v_q,s_q))$ with $\mathrm{Link}^+_F(v_0,\dots,v_q)$. We have a morphism

$$\sigma_{(v_0,s_0),\dots,(v_q,s_q)} : \mathrm{Link}^+_F(v_0,\dots,v_q) \to \mathrm{Link}_{X_0}((v_0,s_0),\dots,(v_q,s_q))$$

defined as follows. Let $(w_0,\dots,w_p) \in \mathrm{Link}^+_F(v_0,\dots,v_q)$. Then $(v_0,\dots,v_q) < (w_0,\dots,w_p)$ so there is a $\phi : \{0,\dots,q\} \to \{0,\dots,p\}$ such that $\phi(i) < \phi(j)$ if $i < j$ and

$w_{\phi(i)} = v_i$ for $i = 0,\ldots,q$. Now take $t_{\phi(i)} = s_i$ for $i = 0,\ldots,q$ and $t_j = s$ if $j \neq \phi(0),\ldots,\phi(q)$, and define

$\sigma_{((v_0,s_0),\ldots,(v_q,s_q))}(w_0,\ldots,w_p) = ((w_0,t_0),\ldots,(w_p,t_p))$.

Now we define a morphism

$$\pi_{((v_0,s_0),\ldots,(v_q,s_q))} : \mathrm{Link}_{X_0}((v_0,s_0),\ldots,(v_q,s_q)) \to \to \mathrm{Link}_F^+(v_0,\ldots,v_q)$$

For $((w_0,t_0),\ldots,(w_p,t_p)) \in \mathrm{Link}_{X_0}((v_0,s_0),\ldots,(v_q,s_q))$, we take $\pi_{((v_0,s_0),\ldots,(v_q,s_q))}((w_0,t_0),\ldots,(w_p,t_p))$ to be the subsequence of $(w_0,\ldots,w_p)$ consisting of these w_i such that either $w_i = v_j$ for some j or $t_i = s$.

A glance at the definitions shows that

$\pi_{((v_0,s_0),\ldots,(v_q,s_q))}\sigma_{((v_0,s_0),\ldots,(v_q,s_q))} = \mathrm{id}_{\mathrm{Link}_F^+(v_0,\ldots,v_q)}$

and for $((w_0,t_0),\ldots,(w_p,t_p)) \in \mathrm{Link}_{X_0}((v_0,s_0),\ldots,(v_q,s_q))$ that

$\sigma_{((v_0,s_0),\ldots,(v_q,s_q))}\pi_{((v_0,s_0),\ldots,(v_q,s_q))}((w_0,t_0),\ldots,(w_p,t_p)) \leqslant ((w_0,t_0),\ldots,(w_p,t_p))$.

By I (2.1)(ii) we conclude that $|\mathrm{Link}_{X_0}((v_0,s_0),\ldots,(v_q,s_q))|$ is homotopy equivalent to $|\mathrm{Link}_F^+(v_0,\ldots,v_q)|$, which is d-q-1-spherical as F is CM.

Take an element $((v_0,s_0),\ldots,(v_p,s_p)) \in F < S >$. Then one easily sees

$$(F < S >)_{((v_0,s_0),\ldots,(v_q,s_q))} \cong F_{(v_0,\ldots,v_q)} < S >$$

Combining this with (5.4) and (6.1) yields

(6.3) <u>Corollary</u>. Let $F \subseteq \mathcal{O}(V)$ be full of dimension d, and let S be a non-empty set. If F is d-dim CM then $F < S >$ is d-dim CM.

III. Acyclicity theorems.

1. Introduction.

The aim of this chapter is to prove that certain posets are spherical, by using the techniques developed in chapter II.

We will show for instance $\mathcal{O}(n)$ is n-1-spherical for all n, and if R is a local ring, then $\mathcal{O}(n,R)$ is n-1-spherical for all n. The proofs are based on building the relevant posets by simple steps from something with known homology.

If R is local, we will show $\mathcal{A}(n,R)$ is n-spherical for all n. We use a different approach here: assuming $\mathcal{A}(m,R)$ is m-spherical for $m < n$, we can show easily $\tilde{H}_i(\mathcal{A}(n,R)) = 0$ if $0 \leqslant i < n-1$. Then we compute generators of $\tilde{H}_{n-1}(\mathcal{A}(n,R))$ which we prove to be zero.

Let R be a subring of $\mathbb{Q}$. By slightly more complicated methods, but basically along the same lines, we prove $\mathcal{O}(n,R)$ is n-1-spherical for all n and $\mathcal{A}(n,R)$ is n-spherical for all n.

2. The homology of $\mathcal{O}(n)$.

This section is devoted to proving

(2.1) Theorem. For all n, the poset $\mathcal{O}(n)$ is n-1- spherical.

Proof. Induction on n. The case $n = 1$ is trivial. Suppose $n \geqslant 2$ is given, and we have shown $\mathcal{O}(k)$ to be k-1-spherical for $k < n$.

We have $\mathcal{O}(n-1) \subset \mathcal{O}(n)$, and we know $\mathcal{O}(n-1)$ to be n-2-spherical. Now define

$$X = \{\vec{v} \in \mathcal{O}(n) \mid (n) \neq \vec{v}\} .$$

Then evidently $\mathcal{O}(n-1) \subset X$; denote by i the inclusion. If we define $\pi : X \to \mathcal{O}(n-1)$ to be the deletion of n, we get a projection satisfying $\pi i = id_{\mathcal{O}(n-1)}$ and $i\pi(\vec{v}) \leqslant \vec{v}$ for $\vec{v} \in X$. By I (2.1)(ii) we find $|i|$ and $|\pi|$ to be homotopy equivalences, hence also X is n-2-spherical.

Now $\mathcal{O}(n)\backslash X = \{(n)\}$ is discrete, hence if we show

(a) $Link_X(n)$ is n-2- spherical

(b) $\tilde{H}_{n-2}(Link_X(n)) \to \tilde{H}_{n-2}X$ is surjective

then II (3.2)(ii) allows us to conclude $\mathcal{O}(n)$ is n-1-spherical. As for (a), we see we can identify $Link_X((n))$ with $Z_0\mathcal{O}(n-1)$ by sending n to z_0. Now for $(v_0,\ldots,v_i) \in \mathcal{O}(n-1)$ we have $\mathcal{O}(n-1)_{(v_0,\ldots,v_i)} \cong \mathcal{O}(n-i-2)$ which is n-i-3-spherical by the induction hypothesis. So II(5.3) yields $Z_0\mathcal{O}(n-1) \cong Link_X((n))$ is n-2-spherical.

To prove (b) note that the following diagram is commutative

$$\begin{array}{ccc} Link_X((n)) & \cong & Z_0\mathcal{O}(n-1) \\ \downarrow & & \downarrow \zeta_0 \\ X & \xrightarrow{\ \pi\ } & \mathcal{O}(n-1) \end{array}$$

As $\tilde{H}_{n-2}(|\zeta_0|)$ is surjective by II(5.3) we have the surjectivity of

$$\tilde{H}_{n-2}(Link_X((n))) \to \tilde{H}_{n-2}X$$

i.e. (b). ⊠

3. The homology of $\mathcal{O}(n,K)$, K a field.

We saw in the last section that the posets $\mathcal{O}(n)$ form a sort of complete family in the sense that $\mathcal{O}(n)_{(v_0,\ldots,v_p)} \cong$ $\cong \mathcal{O}(n-p-1)$. If K is a field, we do not have the analogous property for the $\mathcal{O}(n,K)$. However, we can repair this by considering posets $\mathcal{O}(n,p,K)$, defined as follows. Let $e_1,\ldots,e_n$

be the standard basis of K^n, and define for $p < n$

$$\mathcal{O}(n,p,K) = \mathcal{O}(n,K)_{(e_1,\dots,e_p)}$$

We have $\mathcal{O}(n,0,K) = \mathcal{O}(n,K)$. Furthermore, for all $(v_{p+1},\dots,v_q) \in \mathcal{O}(n,p,K)$ we have

$$\mathcal{O}(n,p,K)_{(v_{p+1},\dots,v_q)} \cong \mathcal{O}(n,q,K)$$

if $p < q < n$, by a change of basis. Of course, $\mathcal{O}(n,p,K)$ is full of dimension $n-p-1$. Observe that $\mathcal{O}(n,p,K) \cong \mathcal{O}(n-p,K)<K^p>$.

(3.1) <u>Theorem</u>. If K is a field, then for all n and p such that $n > p$ the poset $\mathcal{O}(n,p,K)$ is $n-p-1$-spherical.

<u>Proof</u>. Induction on dim $\mathcal{O}(n,p,K) = n-p-1$. If dim $\mathcal{O}(n,p,K) = 0$, i.e. $n = p+1$, then $\mathcal{O}(n,p,K)$ is discrete and non-empty, hence 0-spherical.

Now suppose we have proved the theorem for dim $\mathcal{O}(n',p',K) = n'-p'-1 < d$, with $d \geqslant 1$. So assume we are given an $\mathcal{O}(n,p,K)$ with $n-p-1 = d$. We have to show $\mathcal{O}(n,p,K)$ is d-spherical.

We first want to find a "known piece" of $\mathcal{O}(n,p,K)$. To do this, embed $K^{n-1} \hookrightarrow K^n$ by sending $e_i \mapsto e_i$ if $1 \leqslant i \leqslant n-1$. Then

$$\mathcal{O}(n,p,K) \cap \mathcal{O}(n-1,K) = \mathcal{O}(n-1,p,K) .$$

By the induction hypothesis, $\mathcal{O}(n-1,p,K)$ is $(n-p-2) = (d-1)$-spherical.

Define

$$X_0 = \{(v_0,\dots,v_t) \in \mathcal{O}(n,p,K) | \exists_j v_j \in K^{n-1}\}.$$

Evidently we have an inclusion $\mathcal{O}(n-1,p,K) \subseteq X_0$. Denote this inclusion by i. We want to show $|i|$ is a homotopy equivalence.

The projection

$$\pi : X_0 \to \mathcal{O}(n-1,p,K)$$

is defined as follows. $\pi(v_0,\dots,v_t)$ is the subsequence of $(v_0,\dots,v_t)$ consisting of all v_j such that $v_j \in K^{n-1}$. Evidently π is a morphism of ordered sets, $\pi i = id_{\mathcal{O}(n-1,p,K)}$ and $i\pi(\vec{v}) \leq \vec{v}$ for all $\vec{v} \in X_0$. By I(2.1)(ii), $|i|$ and $|\pi|$ are homotopy equivalences.

So we have X_0 is $d-1 = n-p-2$-spherical. We now glue the rest of $\mathcal{O}(n,p,K)$ to X_0 and show the result is $d = (n-p-1)$-spherical. We glue step by step by defining a filtration on $\mathcal{O}(n,p,K)$. Take for $q = 0,\dots,d+1$

$$X_q = X_0 \cup \{(v_0,\dots,v_t) \mid \forall_j v_j \notin K^{n-1},\ t < q\}.$$

As $d = n-p-1$, $X_{d+1} = \mathcal{O}(n,p,K)$. We see it is sufficient to prove

(3.2) Claim. (i) The poset X_1 is d-spherical.
(ii) If X_{q-1} is d-spherical so is X_q for $q = 2,\dots,d+1$.

So we proceed to prove this.

(i) We have $X_1 \setminus X_0 = \{(v_0) \mid v_0 \notin K^{n-1}\}$. As this is discrete, we can apply II(3.2)(ii) to conclude X_1 is d-spherical if we show
(a) $\mathrm{Link}_{X_0}((v_0))$ is $(d-1) = (n-p-2)$-spherical for $v_0 \notin K^{n-1}$.
(b) $\tilde{H}_{d-1}(\mathrm{Link}_{X_0}((v_0))) \to \tilde{H}_{d-1}(X_0)$ is surjective for $v_0 \notin K^{n-1}$.
First (a). We have

$$\mathrm{Link}_{X_0}((v_0)) = \{(w_0,\dots,w_t) \in X_0 \mid (v_0) < (w_0,\dots,w_t)\}.$$

Since $v_0 \notin K^{n-1}$, sending z_0 to v_0 gives an inclusion $i_{(v_0)} : Z_0\mathcal{O}(n-1,p,K) \to \mathrm{Link}_{X_0}((v_0))$. On the other hand, we have a projection $\pi_{(v_0)} : \mathrm{Link}_{X_0}((v_0)) \to Z_0\mathcal{O}(n-1,p,K)$ defined as follows: if $(w_0,\dots,w_t) \in \mathrm{Link}_{X_0}((v_0))$ first take the subsequence of $(w_0,\dots,w_t)$ consisting of v_0 and the $w_j \in K^{n-1}$, then replace v_0 by z_0. Again $\pi_{(v_0)}$ is a morphism of posets, $\pi_{(v_0)} i_{(v_0)} = id_{Z_0\mathcal{O}(n-1,p,K)}$ and $i_{(v_0)}\pi_{(v_0)}(\vec{w}) \leq \vec{w}$ for all

$\vec{w} \in \mathrm{Link}_{X_0}((v_0))$. By I(2.1)(ii) we conclude $|Z_0\mathcal{O}(n-1,p,K)|$ and $|\mathrm{Link}_{X_0}((v_0))|$ are homotopy equivalent.

If $(w_0,\ldots,w_t) \in \mathcal{O}(n-1,p,K)$, then for $t < n-p-2$ $\mathcal{O}(n-1,p,K)_{(w_0,\ldots,w_t)} \cong \mathcal{O}(n-1,p+t+1,K)$. By the induction hypothesis, this is $(n-1-p-t-1-1) = (d-2-t)$-spherical. If $t = n-p-2$, $\mathcal{O}(n-1,p,K)_{(w_0,\ldots,w_t)} = \phi$, hence -1-spherical. II(5.3) then yields $Z_0\mathcal{O}(n-1,p,K)$ and hence $\mathrm{Link}_{X_0}((v_0))$ is $d-1$-spherical. This settles (a). Now (b). A straightforward computation shows the commutativity of

$$\begin{array}{ccc} \mathrm{Link}_{X_0}((v_0) & \xrightarrow{\pi(v_0)} & Z_0\mathcal{O}(n-1,p,K) \\ \downarrow & & \downarrow \zeta_0 \\ X_0 & \xrightarrow{\pi} & \mathcal{O}(n-1,p,K) \end{array} .$$

The horizontal arrows are homotopy equivalences, and by II(5.3) $\tilde{H}_{d-1}(|\zeta_0|)$ is surjective. Hence (b).

(ii) As $X_q \backslash X_{q-1} = \{(v_0,\ldots,v_{q-1}) \in \mathcal{O}(n,p,K) \mid \forall_j v_j \notin K^{n-1}\}$ is discrete, we can apply II(3.2)(i) if we show that

$$\mathrm{Link}_{X_{q-1}}((v_0,\ldots,v_{q-1}))$$

is $d-1$-spherical for $(v_0,\ldots,v_{q-1}) \in X_q \backslash X_{q-1}$.

In the first place:

$$\mathrm{Link}^-_{X_{q-1}}((v_0,\ldots,v_{q-1})) = \mathrm{Link}^-_{\mathcal{O}(n,p,K)}((v_0,\ldots,v_{q-1})) .$$

Hence by II(4.6) we have

$$|\mathrm{Link}^-_{X_{q-1}}((v_0,\ldots,v_{q-1}))| \cong S^{q-2}$$

So

$$\begin{aligned} |\mathrm{Link}_{X_{q-1}}((v_0,\ldots,v_{q-1}))| &= \mathrm{Hopf}(|\mathrm{Link}^-_{X_{q-1}}(v_0,\ldots,v_{q-1})|, \\ &\qquad |\mathrm{Link}^+_{X_{q-1}}(v_0,\ldots,v_{q-1})|) \\ &\cong \mathrm{Hopf}(S^{q-2},|\mathrm{Link}^+_{X_{q-1}}(v_0,\ldots,v_{q-1})|) \\ &\cong S^{q-1}|\mathrm{Link}^+_{X_{q-1}}((v_0,\ldots,v_{q-1}))| \end{aligned}$$

by I(1.3). So we have to show $\mathrm{Link}^+_{X_{q-1}}(v_0,\dots,v_{q-1})$ is d-q-spherical.

If $q = d+1$, $\mathrm{Link}^+_{X_{q-1}}((v_0,\dots,v_{q-1})) = \phi$ and hence -1-spherical and we are through. So assume $q \leq d$. We have

$$\mathrm{Link}^+_{X_{q-1}}((v_0,\dots,v_{q-1})) = \mathrm{Link}_{X_0}((v_0,\dots,v_{q-1}))$$

from the definition of X_{q-1}. Now for $j = 1,\dots,q-1$ we can choose $\alpha_j \in K$ such that $v'_j = v_j - \alpha_j v_0 \in K^{n-1}$. Also, $< v_0,\dots,v_{q-1} > = < v_0, v'_1,\dots,v'_{q-1} >$ and $< v_0,\dots,v_{q-1} > \cap K^{n-1} = < v'_1,\dots,v'_{q-1} >$. Furthermore we have $(v'_1,\dots,v'_{q-1}) \in \mathcal{O}(n-1,p,K)$. Hence we can form $Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots,v'_{q-1})}$. We want to show the realisation of this is homotopy equivalent to $\mathrm{Link}^+_{X_{q-1}}((v_0,\dots,v_{q-1}))$.

We have an inclusion

$$i_{(v_0,\dots,v_{q-1})} : Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots,v'_{q-1})} \to$$
$$\to \mathrm{Link}_{X_0}((v_0,\dots,v_{q-1})),$$

by substituting $v_0,\dots,v_{q-1}$ for $z_0,\dots,z_{q-1}$. On the other hand

$$\pi_{(v_0,\dots,v_{q-1})}: \mathrm{Link}_{X_0}((v_0,\dots,v_{q-1})) \to$$
$$\to Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots,v'_{q-1})}$$

is defined by: $\pi_{(v_0,\dots,v_{q-1})}(w_0,\dots,w_t)$ is the subsequence of $(w_0,\dots,w_t) \in \mathrm{Link}_{X_0}((v_0,\dots,v_{q-1}))$ consisting of $v_0,\dots,v_{q-1}$ and the $w_j \in K^{n-1}$ with v_j replaced by z_j. It is an easy exercise in linear algebra that this lands in $Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots v'_{q-1})}$. As before

$$\pi_{(v_0,\dots,v_{q-1})} i_{(v_0,\dots,v_{q-1})} = \mathrm{id}_{Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots,v'_{q-1})}}$$

and for $\vec{w} \in \mathrm{Link}_{X_0}((v_0,\dots,v_{q-1}))$ we have $i_{(v_0,\dots,v_{q-1})}\pi_{(v_0,\dots,v_{q-1})}(\vec{w}) \leq \vec{w}$. Hence by I(2.1)(ii)

$$|\mathrm{Link}_{X_0}((v_0,\dots,v_{q-1}))| \cong |Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\dots,v'_{q-1})}|.$$

So it remains to show $Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\ldots,v'_{q-1})}$ is d-q-spherical. But we have

$$Z_{q-1}\mathcal{O}(n-1,p,K)_{(v'_1,\ldots,v'_{q-1})} \cong Z_{q-1}\mathcal{O}(n-1,p+q-1,K)$$

Once more, we see from the induction hypothesis (stated at the beginning of the proof of (3.1)) and II(5.3) this is $n-1-p-q+1-1 = d-q$-spherical. This finishes the proof of (3.2) and hence of theorem (3.1).

From II(5.4) it follows we can reformulate our result as follows.

(3.3) Corollary. If K is a field, then for all n the poset $\mathcal{O}(n,K)$ is n-1-dim CM.

4. Euclidean rings.

In this section we wish to extend the result of section 3 to a larger class of rings. As our reasoning was partly geometrical, we wish to consider rings that allow certain analogous geometrical arguments. A good choice appears to be Euclidean rings. We give a definition first.

(4.1) Definition. A commutative ring R is called Euclidean if we have a function $\mu : R \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that

(i) $\mu(a) = 0$ if and only if $a = 0$, $\mu(1) = 1$.

(ii) if $a,b \neq 0$ then $\mu(ab) \geqslant \mu(b)$.

(iii) There are functions $\kappa,\rho : R \times R\setminus\{0\} \to R$ with the properties: for all $a \in R$, $b \in R\setminus\{0\}$

$$a = \kappa(a,b)b + \rho(a,b)$$

$$\mu(\rho(a,b)) < \mu(b) .$$

Furthermore, if $\mu(a) < \mu(b)$ then $\kappa(a,b) = 0$, $\rho(a,b) = a$.

We first give some properties. If b is a unit, then $\mu(b) = 1$; on the other hand, if $\mu(b) = 1$, then $1 = \kappa(1,b)b + \rho(1,b)$ with $\mu(\rho(1,b)) = 0$ so $\rho(1,b) = 0$ and $b^{-1} = \kappa(1,b)$, i.e. b is a unit. A Euclidean ring is a principal ideal domain.

Examples. i) If K is a field, define $\mu(0) = 0$, $\mu(\lambda) = 1$ if $\lambda \neq 0$.

ii) If $R = \mathbb{Z}$ define $\mu(z) = |z|$.

iii) If R is a subring of $\mathbb{Q}$, then $R = \mathbb{Z}[\frac{1}{p} | p \in S]$ for some set of prime numbers S. An element $r \in \mathbb{Q}$ can be written as $r = r_S r_S'$ where r_S is a product of prime numbers in S or their inverses, and $\pm r_S'$ is a product of prime numbers not in S or their inverses. We define $|r|_S = |r_S'|$, and we have $|rt|_S = |r|_S |t|_S$ for all $r,t \in \mathbb{Q}$.

For $r \in \mathbb{Q}\setminus\{0\}$ we have $r \in R$ if and only if $r_S' \in \mathbb{Z}$. Define $\mu(r) = |r|_S$ for $r \in R$. We claim $|\cdot|_S$ makes R into a Euclidean ring. So let $a,b \in R$. Write $a = a_S a_S'$, $b = b_S b_S'$ as above. Then we have by division in $\mathbb{Z}$

$$a_S' = q b_S' + r$$

with $|r| < |b_S'|$. Now we have

$$a = (a_S b_S^{-1} q) b + a_S r$$

As $|a_S r|_S = |r|_S \leqslant |r| \leqslant |b_S'| = |b|_S$ we are through.

iv) If $R = \mathbb{Z}[\sqrt{2}]$ define $\mu(a+b\sqrt{2}) = |a^2-2b^2|$.

v) If $R = K[T]$, the ring of polynomials over a field K, then define $\mu(f) = \deg(f) + 1$, $\mu(0) = 0$.

In cases ii), iii), iv) and v), the division algorithm yields the existence of κ and ρ.

If R is Euclidean, then each finitely generated projective over R is free, so a sequence $(v_0,\ldots,v_q)$ of elements in R^n is an element of $\mathcal{O}(n,R)$ if and only if $< v_0,\ldots,v_q >$ is a direct summand of R^n and rank $< v_0,\ldots,v_q > = q+1$.

For a given ring R, denote by $e_1,\ldots,e_n$ the unit vectors in R^n. As in the field case, define $\mathcal{O}(n,p,R) = \mathcal{O}(n,R)_{(e_1,\ldots,e_p)}$ for $p < n$. Write $\mathcal{O}(n,0,R) = \mathcal{O}(n,R)$. For any R, $\mathcal{O}(n,p,R)$ is full of dimension n-p-1. Also, for $p < q < n$ we have if $(v_{p+1},\ldots,v_q) \in \mathcal{O}(n,p,R)$ that

$$\mathcal{O}(n,p,R)_{(v_{p+1},\ldots,v_q)} \cong \mathcal{O}(n,q,R)$$

We can now state the analogue of (3.1) for Euclidean rings. Observe it is in fact an extension of (3.1) because of example i).

(4.2) Theorem. Let R be a Euclidean ring. For all n and p such that $p < n$, the poset $\mathcal{O}(n,p,R)$ is n-p-1-spherical.

Proof. Induction on dim $\mathcal{O}(n,p,R) = n-p-1$. For $n-p-1 = 0$, $\mathcal{O}(n,p,R)$ is discrete and non-empty, hence 0-spherical.

Now suppose we have shown $\mathcal{O}(n',p',R)$ is n'-p'-1-spherical for $n'-p'-1 < d$ with $d \geq 1$. Then we have to show if $n-p-1 = d$ that $\mathcal{O}(n,p,R)$ is d-spherical.

First we define an infinite filtration on $\mathcal{O}(n,p,R)$. To do this, we look at the last (n^{th}) coordinate of vectors in R^n. If $v \in R^n$, we shall write its coordinates as $v^{(1)},\ldots,v^{(n)}$, and we use $\mu(v^{(n)})$ to measure v. Now define for $q \geq 0$

$$X_q = \{(v_0,\ldots,v_t) \in \mathcal{O}(n,p,R) \mid \exists_j \mu(v_j^{(n)}) \leq q\}.$$

Obviously the X_q are subposets of $\mathcal{O}(n,p,R)$ such that $X_q \subseteq X_{q+1}$

for $q \geq 0$, and

$$\varinjlim X_q = \sigma(n,p,R).$$

(4.3) <u>Claim</u>. i) X_1 is d-spherical.

ii) If for $q \geq 1$, X_q is d-spherical, so is X_{q+1}.

Taking this claim for granted, we can easily show $\sigma(n,p,R)$ is d-spherical. In fact, because

$$\sigma(n,p,R) = \varinjlim X_q$$

we have

$$|\sigma(n,p,R)| = \varinjlim |X_q|$$

and hence

$$\tilde{H}_* |\sigma(n,p,R)| = \varinjlim \tilde{H}_* |X_q|$$

so indeed our claim (4.3) yields $\sigma(n,p,R)$ is d-spherical.

Before proving (4.3) we first want to make some preparations. Define for $q \geq 0$

$$\sigma_q = \{(v_0,\ldots,v_t) \in \sigma(n,p,R) \mid \forall_j \mu(v_j^{(n)}) \leq q\}$$

and for $q \geq 1$

$$U_q = X_{q-1} \cup \sigma_q$$

We take $U_0 = \sigma_0$. We have inclusions

$$\begin{array}{ccc} \sigma_q & \xhookrightarrow{\;i_q\;} & X_q \\ & {\scriptstyle i'_q} \searrow \quad \nearrow {\scriptstyle i''_q} & \\ & U_q & \end{array}$$

which satisfy

(4.4) For all $q \geq 0$, $|i_q|$, $|i'_q|$ and $|i''_q|$ are homotopy equivalences.

To prove this, define $\pi_q : X_q \to \mathcal{O}_q$ as follows: $\pi_q(v_0,\ldots,v_t)$ is the subsequence of $(v_0,\ldots,v_t) \in X_q$ consisting of the v_j such that $\mu(v_j^{(n)}) \leq q$. Then π_q is a morphism of ordered sets, $\pi_q i_q = id_{\mathcal{O}_q}$, and $i_q \pi_q(\vec{v}) \leq \vec{v}$ for all $\vec{v} \in X_q$. Hence by I(2.1)(ii), $|\pi_q|$ is a homotopy inverse to $|i_q|$.

Defining $\pi'_q = \pi_q|U_q$, we have of course that $|\pi'_q|$ is a homotopy inverse to $|i'_q|$. As the diagram above commutes, $|i''_q|$ is also a homotopy equivalence.

Now we prove (i) and (ii) of claim (4.3).

i) We have $\mathcal{O}_0 = \mathcal{O}(n-1,p,R)$ and $|\mathcal{O}_0| \cong |X_0|$ by (4.4). By the induction hypothesis, X_0 is d-1-spherical. Proceeding as in the proof of (3.1), we find that U_1 is d-spherical, because a is a unit in R if and only if $\mu(a) = 1$. Then by (4.4) we conclude X_1 is d-spherical too.

ii) Suppose we know X_q is d-spherical. As $|X_{q+1}| \cong |U_{q+1}|$, we only have to show U_{q+1} is d-spherical. We define

$$Y_r = X_q \cup (\mathcal{O}_{q+1})_{\leq r-1}$$

We have a filtration on U_{q+1}

$$X_q = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_{d+1} = U_{q+1} .$$

We already know X_q is d-spherical so we have to show: if Y_r is d-spherical so is Y_{r+1}.

We obviously have

$$Y_{r+1} \backslash Y_r = \{(v_0,\ldots,v_r) \in \mathcal{O}(n,p,R) | \forall_j \mu(v_j^{(n)}) = q+1\}.$$

This is discrete, so we can apply II(3.2)(i) to show Y_{r+1} is d-spherical if we show $\mathrm{Link}_{Y_r}(v_0,\ldots,v_r)$ is d-1-spherical for $(v_0,\ldots,v_r) \in \mathcal{O}(n,p,R)$ such that $\mu(v_j^{(n)}) = q+1$ for all j.

Now

$$\mathrm{Link}^-_{Y_r}(v_0,\ldots,v_r) = \mathrm{Link}^-_{\mathcal{O}(n,p,R)}(v_0,\ldots,v_r)$$

so by II(4.6)

$$|\mathrm{Link}^-_{Y_r}(v_0,\ldots,v_r)| \cong S^{r-1}$$

Hence

$$\begin{aligned}|\mathrm{Link}_{Y_r}(v_0,\ldots,v_r)| &= \mathrm{Hopf}(|\mathrm{Link}^-_{Y_r}(v_0,\ldots,v_r)|,|\mathrm{Link}^+_{Y_r}(v_0,\ldots,v_r)|)\\ &= \mathrm{Hopf}(S^{r-1},|\mathrm{Link}^+_{Y_r}(v_0,\ldots,v_r)|)\\ &= S^r|\mathrm{Link}^+_{Y_r}(v_0,\ldots,v_r)| .\end{aligned}$$

Now for $r = d$, we have $\mathrm{Link}^+_{Y_r}(v_0,\ldots,v_r) = \phi$, so $|\mathrm{Link}_{Y_r}(v_0,\ldots,v_r)| = S^{d-1}$ i.e. $\mathrm{Link}_{Y_r}(v_0,\ldots,v_r)$ is d-1-spherical. Hence we can assume $r < d$. We have to prove then

$$\mathrm{Link}^+_{Y_r}(v_0,\ldots,v_r) = \mathrm{Link}_{X_q}(v_0,\ldots,v_r)$$

is d-r-1-spherical.

We are going to break down $\mathrm{Link}_{X_q}(v_0,\ldots,v_r)$ for $r < d$ step by step. Define

$$\mathcal{O}_{q,(v_0,\ldots,v_r)}=\{(w_0,\ldots,w_t) \in \mathcal{O}(n,p,R)_{(v_0,\ldots,v_r)}|\forall_j\mu(w_j^{(n)}) \leq q\}$$

$$X_{q,(v_0,\ldots,v_r)}=\{(w_0,\ldots,w_t) \in \mathcal{O}(n,p,R)_{(v_0,\ldots,v_r)}|\exists_j\mu(w_j^{(n)}) \leq q\}$$

In fact we have

$$\mathcal{O}_{q,(v_0,\ldots,v_r)}= \mathcal{O}(n,p,R)_{(v_0,\ldots,v_r)} \cap \mathcal{O}_q$$
$$X_{q,(v_0,\ldots,v_r)}= \mathcal{O}(n,p,R)_{(v_0,\ldots,v_r)} \cap X_q .$$

Sending $z_0,\ldots,z_r$ to $v_0,\ldots,v_r$ we have

$$\mathrm{Link}_{X_q}(v_0,\ldots,v_r) = Z_r(X_{q,(v_0,\ldots,v_r)})$$

The inclusion $\mathcal{O}_{q,(v_0,\ldots,v_r)} \subset X_{q,(v_0,\ldots,v_r)}$ yields an inclusion morphism

$$i : Z_r(\mathcal{O}_{q,(v_0,\ldots,v_r)}) \to Z_r(X_{q,(v_0,\ldots,v_r)})$$

whereas $\pi_q|X_{q,(v_0,\ldots,v_r)}$ defines a projection $X_{q,(v_0,\ldots,v_r)} \to \mathcal{O}_{q,(v_0,\ldots,v_r)}$. This yields a projection

$$\pi : Z_r(X_{q,(v_0,\ldots,v_r)}) \to Z_r(\mathcal{O}_{q,(v_0,\ldots,v_r)}) .$$

In fact, for $(w_0,\dots,w_t) \in Z_r(X_{q,(v_0,\dots,v_r)})$, we have that $\pi(w_0,\dots,w_t)$ is the subsequence of $(w_0,\dots,w_t)$ consisting of the w_j such that $\mu(w_j^{(n)}) \leq q$, and $z_0,\dots,z_r$. We see π is a morphism of ordered sets such that $\pi i = \mathrm{id}_{Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)})}$ and $i\pi(\vec{w}) \leq \vec{w}$ for all $\vec{w} \in Z_r(X_{q,(v_0,\dots,v_r)})$. As usual, I(2.1) (i) yields $|i|$ and $|\pi|$ are homotopy equivalences. It remains to show $Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)})$ is d-r-1-spherical.

The inclusion

$$\phi : Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)}) \to Z_r(\mathcal{O}(n,p,R)_{(v_0,\dots,v_r)})$$

admits a section

$$\psi : Z_r(\mathcal{O}(n,p,R)_{(v_0,\dots,v_r)}) \to Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)})$$

which we shall define in the following. If $(w_0,\dots,w_t) \in \mathcal{O}(n,p,R)_{(v_0,\dots,v_r)}$, then $(v_0,\dots,v_r,w_0,\dots,w_t) \in$ $\in \mathcal{O}(n,p,R)$ and hence

$$\psi_0(w_0,\dots,w_t) = (w_0-\kappa(w_0^{(n)},v_0^{(n)})v_0,\dots,w_t-\kappa(w_t^{(n)},v_0^{(n)})v_0)$$

is an element of $\mathcal{O}(n,p,R)$. However, we have $w_j^{(n)} = \kappa(w_j^{(n)},v_0^{(n)})v_0^{(n)} + \rho(w_j^{(n)},v_0^{(n)})$ with $\mu(\rho(w_j^{(n)},v_0^{(n)}) <$ $< \mu(v_0^{(n)}) = q+1$. As $\rho(w_j^{(n)},v_0^{(n)})$ is the last coordinate of $w_j - \kappa(w_j^{(n)},v_0^{(n)})v_0$ we have $\psi_0(w_0,\dots,w_t) \in \mathcal{O}_{q,(v_0,\dots,v_r)}$. Hence ψ_0 is a map $\mathcal{O}(n,p,R)_{(v_0,\dots,v_r)} \to \mathcal{O}_{q,(v_0,\dots,v_r)}$. If $\mu(w_j^{(n)}) \leq q$, then $\kappa(w_j^{(n)},v_0^{(n)}) = 0$, so $\psi_0|\mathcal{O}_{q,(v_0,\dots,v_r)} =$ $= \mathrm{id}_{\mathcal{O}_{q,(v_0,\dots,v_r)}}$. Also, ψ_0 is a morphism of posets.

One easily sees ψ_0 lifts uniquely to a morphism

$$\psi : Z_r(\mathcal{O}(n,p,R)_{(v_0,\dots,v_r)}) \to Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)})$$

which satisfies $\psi|Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)}) = \mathrm{id}$, hence $\psi\phi = \mathrm{id}$ and $\tilde{H}_*(\psi)\tilde{H}_*(\phi) = \mathrm{id}$. We find $Z_r(\mathcal{O}_{q,(v_0,\dots,v_r)})$ is d-r-1-spherical if $Z_r(\mathcal{O}(n,p,R)_{(v_0,\dots,v_r)})$ is. But the latter poset is isomorphic to $Z_r(\mathcal{O}(n,p+r+1,R))$. From the induction hypothesis and II(5.3) we conclude this last poset is $n-p-r-1-1 = d-r-1$-spherical. We have proved claim (4.3) and hence also theorem (4.2). ⊠

Analogous to (3.3) we have

(4.5) Corollary. If R is a Euclidean ring, then the poset $\mathcal{O}(n,R)$ is n-1-dim CM.

5. Affine geometry.

The aim of this section is to prove for a field K that $\mathcal{A}(n,K)$ is n-spherical and even n-dim CM, and that the same holds for $\mathcal{A}(n,R)$ if R is a subring of $\mathbb{Q}$.

This section contains an introductory part in which we consider Tits buildings and the like, and the proof proper.

We start with some remarks on the homology of skeletons of simplices. As usual Δ_p is the ordered set of positive integers $1,\ldots,p+1$. Its realisation is the topological p-simplex Δ^p. A non-degenerate k-simplex corresponds to a sequence $i_0 < \ldots < i_k$. We denote this simplex by $\Delta(i_0,\ldots,i_k)$ when considered as an element of $\mathcal{C}_k(\Delta^p)$. If $i_s = i_{s+1}$ for some s, then $i_0 \leq \ldots \leq i_k$ corresponds to a degenerate k-simplex $\Delta(i_0,\ldots,i_k)$ which is zero in $\mathcal{C}_k(\Delta^p)$. If σ is a permutation of $0,\ldots,k$, then $\Delta(i_{\sigma 0},\ldots,i_{\sigma k}) = \varepsilon(\sigma)\Delta(i_0,\ldots,i_k)$ in $\mathcal{C}_k(\Delta^p)$. This corresponds to a change of orientation. For any $i_0,\ldots,i_k \in \{1,\ldots,p+1\}$, we denote the boundary of $\Delta(i_0,\ldots,i_k)$ by $\gamma(i_0,\ldots,i_k)$, i.e.

$$\gamma(i_0,\dots,i_k) = \sum_{j=0}^{k} (-1)^j \Delta(i_0,\dots,\hat{i}_j,\dots,i_k) .$$

Obviously for $\sigma \in S_{k+1}$ we have $\gamma(i_{\sigma 0},\dots,i_{\sigma k}) = \varepsilon(\sigma)\gamma(i_0,\dots i_k)$

Because $\delta^2 = 0$, we have for $i_0,\dots,i_k \in \{1,\dots,p+1\}$

$$\sum_{j=0}^{k} (-1)^j \gamma(i_0,\dots,\hat{i}_j,\dots,i_k) = 0 .$$

Taking $i_0 = i$, and, for $j = 1,\dots,k$ observing that

$$\gamma(i,i_1,\dots,i_{j-1},i_{j+1},\dots,i_k)=(-1)^{j-1}\gamma(i_1,\dots,i_{j-1},i,i_{j+1},\dots,i_k)$$

we conclude

$$\gamma(i_1,\dots,i_k) = \sum_{j=1}^{k} \gamma(i_1,\dots,i_{j-1},i\ ,i_{j+1},\dots,i_k)$$

Obviously we have

$$\mathcal{O}(N_*\Delta_p) = \{(i_0,\dots,i_k) | 1 \leqslant i_0 < \dots < i_k \leqslant p+1\}$$

So $\mathcal{O}(N_*\Delta_p)$ can be identified with the set of all non-empty subsets of $\{1,\dots,p+1\}$, ordered by inclusion. By II §4, $|\mathcal{O}(N_*\Delta_p)|$ is the first barycentric subdivision of Δ^p. We have an inclusion of chain complexes $\mathcal{C}_*(\Delta^p) \to \mathcal{C}_*(\mathcal{O}(N_*\Delta_p))$, and we identify $\mathcal{C}_*(\Delta^p)$ to its image in $\mathcal{C}_*(\mathcal{O}(N_*\Delta_p))$. We find

$$\Delta(i_0,\dots,i_k)= \sum_{\sigma\in S_{k+1}} \varepsilon(\sigma)\{i_{\sigma 0}\} \subseteq \dots \subseteq \{i_{\sigma 0},\dots,i_{\sigma k}\}$$

and so

$$\gamma(i_0,\dots,i_k)= \sum_{j=0}^{k} (-1)^j \Delta(i_0,\dots,\hat{i}_j,\dots,i_k)$$

$$= \sum_{j=0}^{k} (-1)^j \sum_{\substack{\sigma\in S_{k+1}\\ \sigma j = j}} \varepsilon(\sigma)\{i_{\sigma 0}\} \subseteq \dots \subseteq \{i_{\sigma 0},\dots,\hat{i}_{\sigma j},\dots,i_{\sigma k}\}$$

Let $\sigma \in S_{k+1}$ be such that $\sigma j = j$ and take $\tau = \sigma\circ(j,\dots,k)$. Then

$$\varepsilon(\sigma)\{i_{\sigma 0}\} \subseteq \dots \subseteq \{i_{\sigma 0},\dots\hat{i}_{\sigma j},\dots,i_{\sigma k}\} =$$
$$=(-1)^{k-j}\varepsilon(\tau)\{i_{\tau 0}\} \subseteq \dots \subseteq\{i_{\tau 0},\dots,i_{\tau(k-1)}\}$$

whence

$$\gamma(i_0,\dots,i_k)=(-1)^k \sum_{\sigma\in S_{k+1}} \varepsilon(\sigma)\{i_{\sigma 0}\} \subseteq\dots\subseteq \{i_{\sigma 0},\dots,i_{\sigma(k-1)}\}.$$

For $0 \leqslant k \leqslant p$, the k-skeleton Δ^p_k of Δ^p is a homology wedge of k-spheres, the homology being generated by the bounderies of the k+1-simplices in Δ^p, i.e. by the images of the $\gamma(i_0,\ldots,i_{k+1})$ for $i_0,\ldots,i_{k+1} \in \{1,\ldots,p+1\}$. We denote this image by $\gamma(i_0,\ldots,i_{k+1})$, too. By II, §4 we see that $|\mathcal{O}(N_*\Delta_p)_{\leqslant k}|$ is the first barycentric subdivision of Δ^p_k. We conclude

(5.1) <u>Lemma</u>. Let p be a non-negative integer, $0 \leqslant l \leqslant p$.

(i) $\mathcal{O}(N_*\Delta_p)_{\leqslant l}$ is l-spherical, $\tilde{H}_l(\mathcal{O}(N_*\Delta_p)_{\leqslant l})$ being generated by

$$\gamma(i_0,\ldots,i_{l+1})=(-1)^{l+1}\sum_{\sigma\in S_{l+1}} \varepsilon(\sigma)\{i_{\sigma 0}\}\subseteq\ldots\subseteq\{i_{\sigma 0},\ldots,i_{\sigma l}\}$$

for $i_0,\ldots,i_{l+1} \in \{1,\ldots,p+1\}$. Moreover, if $i_s = i_t$ for some $s \neq t$, then $\gamma(i_0,\ldots,i_{l+1}) = 0$ in $\tilde{H}_l(\mathcal{O}(N_*\Delta_p)_{\leqslant l})$.

(ii) For $i_1,\ldots,i_{l+2}$, $i \in \{1,\ldots,p+1\}$ we have

$$\gamma(i_1,\ldots,i_{l+2})=\sum_{j=1}^{l+2}\gamma(i_1,\ldots,i_{j-1},i,i_{j+1},\ldots,i_{l+2}).$$

For the moment, take R to be a Euclidean ring with field of fractions K. We define the Tits building of R^n by

$$T(n,R) = \{V \subset R^n \mid R^n/V \text{ free, } V \neq 0,R^n\}$$

for $n \geqslant 2$, and $T(1,R) = \phi$. The affine Tits building of R^n is defined by

$$A(n,R) = \{a+V \mid a \in R^n,\ R^n/V \text{ free, } V \neq R^n\}$$

for $n \geqslant 1$, and $A(0,R) = \phi$. Both T(n,R) and A(n,R) are partially ordered by inclusion.

One readily sees that T(n,R) and A(n,R) are non-empty if $n \geqslant 2$ and $n \geqslant 1$, respectively, and they are homogeneous of dimension n-2 and n-1, respectively.

We can consider R^n as a lattice in K^n using the standard bases $e_1,\ldots,e_n$. Hence we have a map

$$\tau(n,R) : T(n,R) \to T(n,K)$$

sending $V \mapsto KV$, which is an isomorphism as the reader can prove easily, since R is Euclidean and hence a P.I.D.

For points $v_0,\dots,v_k \in R^n$ we define

$$A(v_0,\dots,v_k) = v_0 + (K < v_1-v_0,\dots,v_k-v_0> \cap R^n)$$

Evidently $A(v_0,\dots,v_k) \in A(n,R) \cup \{R^n\}$, and if $a+V \in A(n,R)$ is such that $v_0,\dots,v_k \in a+V$, then we have $A(v_0,\dots,v_k) \subset a+V$.

Our first result concerns the homology of T(n,R) and A(n,R). This result is well-known, see for example Quillen [9], Lusztig [4]. Our proof is based on Lusztig's proof.

(5.2) Proposition. Let R be a Euclidean ring with field of fractions K.

(i) For $n \geq 1$, T(n,R) is n-2-spherical. Moreover, for $n \geq 2$, $\tilde{H}_{n-2}(T(n,R))$ is generated by elements

$$g(L_1,\dots,L_n) = \sum_{\sigma\in S_n} \varepsilon(\sigma) L_{\sigma 1} \subseteq K(L_{\sigma 1}+L_{\sigma 2}) \cap R^n \subseteq \dots$$

$$\dots \subseteq K(L_{\sigma 1} + \dots + L_{\sigma(n-1)}) \cap R^n$$

for lines $L_1,\dots,L_n$ in T(n,R). If rank $(\Sigma L_i) < n$, then $g(L_1,\dots,L_n) = 0$.

(ii) For $n \geq 0$, A(n,R) is n-1-spherical. For $n \geq 1$, the group $\tilde{H}_{n-1}(A(n,R))$ is generated by elements

$$g(v_0,\dots,v_n) = \sum_{\sigma\in S_{n+1}} \varepsilon(\sigma) A(v_{\sigma 0}) \subseteq \dots \subseteq A(v_{\sigma 0},\dots,v_{\sigma(n-1)})$$

for $v_0,\dots,v_n \in R^n$. If rank $<v_1-v_0,\dots,v_n-v_0> < n$ i.e. $v_0,\dots,v_n$ not in general position in K^n, then $g(v_0,\dots,v_n) = 0$.

Proof. (i) As $\tau(n,R)$ is an isomorphism, we may assume we work with a field K. Assume $n \geq 2$.

Define

$$\tilde{T}(n,K) = \{\{L_1,\dots,L_k\} \mid L_i \in T(n,K),\ \dim L_i = 1,\ L_i \neq L_j \text{ for } i \neq j,\ \Sigma L_i \neq K^n\}$$

and order it by inclusion. Define

$$\nu : \widetilde{T}(n,K) \to T(n,K), \{L_1,\ldots,L_k\} \mapsto \Sigma L_i .$$

We see ν is a morphism of ordered sets. For $V \in T(n,K)$ we have

$$\nu/V = \{\{L_1,\ldots,L_k\} \in \widetilde{T}(n,K) \mid \forall_i L_i \subset V\}.$$

We see that any finite subset of ν/V has a supremum. It follows that $|\nu/V|$ is contractible. By theorem I (2.2) we conclude $|\nu|$ is a homotopy equivalence.

Now $T(n,R)$ is n-2-dimensional, so we must prove $\widetilde{H}_i(\widetilde{T}(n,K)) = 0$ for $i \leqslant n-3$. We clearly have to show: given a finite number of elements in $\widetilde{T}(n,K)$, then they are contained in a subset of $\widetilde{T}(n,K)$ which has vanishing homology in dimensions $\leqslant n-3$.

A finite subset of $\widetilde{T}(n,K)$ is clearly contained in a subset of the following type. Take distinct lines $L_0,\ldots,L_p$ in K^n, and define

$$\widetilde{T}_{\{L_0,\ldots,L_p\}} = \{\{L_{i_1},\ldots,L_{i_k}\} \in \widetilde{T}(n,K) \mid \forall_j L_{i_j} \in \{L_0,\ldots,L_p\}\}$$

Take

$$\Delta_{\{L_0,\ldots,L_p\}} = \{\{L_{i_1},\ldots,L_{i_k}\} \mid \forall_j L_{i_j} \in \{L_0,\ldots,L_p\}\}.$$

Evidently

$$\Delta_{\{L_0,\ldots,L_p\}} \cong \mathcal{O}(N_*\Delta_p)$$

by sending $(i_1,\ldots,i_k)$, with $i_l < i_m$ if $l < m$ to $\{L_{i_1},\ldots,L_{i_k}\}$. The inclusion $\widetilde{T}_{\{L_0,\ldots,L_p\}} \to \Delta_{\{L_0,\ldots,L_p\}}$ identifies $\widetilde{T}_{\{L_0,\ldots,L_p\}}$ with a subset of $\mathcal{O}(N_*\Delta_p)$ having the chain property. Since for $k < n$ any $\{L_{i_1},\ldots,L_{i_k}\} \in \Delta_{\{L_0,\ldots,L_p\}}$ is an element of $\widetilde{T}_{\{L_0,\ldots,L_p\}}$, we see $|\widetilde{T}_{\{L_0,\ldots,L_p\}}|$ can be identified with a subcomplex of Δ^p containing the n-2-skeleton, cf. II(4.3), II (4.4), hence has zero homology in dimensions $\leqslant n-3$.

As each finite subset of $\widetilde{T}(n,K)$ is contained in some

$\widetilde{T}_{\{L_0,\ldots,L_p\}}$, the group $\widetilde{H}_{n-2}(\widetilde{T}(n,K))$ and hence also $\widetilde{H}_{n-2}(T(n,K))$ is covered by the images of the $\widetilde{H}_{n-2}(\widetilde{T}_{\{L_0,\ldots,L_p\}})$. For each set of distinct lines $L_0,\ldots,L_p$ we have a surjection

$$\widetilde{H}_{n-2}((\Delta_{\{L_0,\ldots,L_p\}})_{\leq n-2}) \twoheadrightarrow \widetilde{H}_{n-2}(\widetilde{T}_{\{L_0,\ldots,L_p\}})$$

because of the above. Applying (5.1)(i) we see that $\widetilde{H}_{n-2}(\widetilde{T}_{\{L_0,\ldots,L_p\}})$ is generated by elements

$$\overline{\gamma}(M_1,\ldots,M_n)=(-1)^{n-1}\sum_{\sigma\in S_n}\varepsilon(\sigma)\{M_{\sigma 1}\}\subseteq\ldots\subseteq\{M_{\sigma 1},\ldots,M_{\sigma(n-1)}\}$$

with $M_1,\ldots,M_n \in \{L_0,\ldots,L_p\}^n$. Now if $\Sigma M_i \neq K^n$, then $\overline{\gamma}(M_1,\ldots,M_n)$ is sent to an element of

$$\mathrm{im}(\widetilde{H}_{n-2}(\nu/\Sigma M_i) \to \widetilde{H}_{n-2}(T(n,R)))$$

which is zero because $|\nu/\Sigma M_i|$ is contractible. As the image of $\overline{\gamma}(M_1,\ldots,M_n)$ in $\widetilde{H}_{n-2}(T(n,R))$ is equal to $(-1)^{n-1}g(M_1,\ldots,M_n)$, this settles (i).

(ii) This is proved analogously, by defining

$$\widetilde{A}(n,R) = \{\{v_0,\ldots,v_k\}\mid v_i \in R^n, v_i \neq v_j \text{ if } i \neq j,\ A(v_0,\ldots,v_k) \neq R^n\}$$

and

$$\zeta : \widetilde{A}(n,R) \to A(n,R)$$

by $\zeta(\{v_0,\ldots,v_k\}) = A(v_0,\ldots,v_k)$.

<u>Remark</u>. It is clear that it follows from (5.1)(ii)that the $g(L_1,\ldots,L_n)$ satisfy the relation

$$g(L_1,\ldots,L_n) = \sum_{i=1}^{n} g(L_1,\ldots,L_{i-1},L,L_{i+1},\ldots,L_n)$$

for lines $L_1,\ldots,L_n$, $L \in T(n,R)$ and in $\widetilde{H}_{n-1}(A(n,R))$ we have the relation

$$g(v_0,\ldots,v_n) = \sum_{i=0}^{n} g(v_0,\ldots,v_{i-1},v,v_{i+1},\ldots,v_n)$$

for $v_0,\ldots,v_n$, $v \in R^n$.

For $V \in T(n,R)$ we can take a complement W such that

$V \oplus W = R^n$. Then the map

$$T(\dim W,R) \to \mathrm{Link}^+_{T(n,R)}(V)$$

given by $U \to V \oplus U$ for $U \subset W$, is an isomorphism, hence (i) of

(5.3) Lemma. i) For $V \in T(n,R)$ we have

$$\mathrm{Link}^+_{T(n,R)}(V) \cong T(n-\dim V,R)$$

ii) For $a+V \in A(n,R)$ we have

$$\mathrm{Link}^+_{A(n,R)}(a+V) \cong T(n-\dim V,R) .$$

Proof: ii) For $V = 0$, we have by taking a as origin

$$\mathrm{Link}^+_{A(n,R)}(0) = T(n,R)$$

and for $\dim V > 0$ we find, again by taking a as origin

$$\mathrm{Link}^+_{A(n,R)}(V) \cong \mathrm{Link}^+_{T(n,R)}(V)$$
$$\cong T(n-\dim V,R)$$

by i).

Next we consider $\mathcal{O}(n,R)_{\leq n-2}$. By theorem (4.2), $\mathcal{O}(n,R)$ is n-1-spherical, hence by II(4.4)(i), $\mathcal{O}(n,R)_{\leq n-2}$ is n-2-spherical. We define

$$\phi(n,R) : \mathcal{O}(n,R)_{\leq n-2} \to T(n,R)$$

by $\phi(n,R)(v_0,\ldots,v_k) = < v_0,\ldots,v_k >$. We have for $(v_0,\ldots,v_k) \in \mathcal{O}(n,R)_{\leq n-2}$ that

$$ht_{T(n,R)}\phi(n,R)(v_0,\ldots,v_k) = \dim < v_0,\ldots,v_k > -1$$
$$= ht_{\mathcal{O}(n,R)_{\leq n-2}}(v_0,\ldots,v_k) .$$

Furthermore, for an arbitrary $V \in T(n,R)$

$$\phi(n,R)/V \cong \mathcal{O}(\dim V,R) .$$

It is a homogeneous poset of dimension $\dim V-1 = ht_{T(n,R)}(V)$. It follows $\phi(n,R)$ is a homogeneous morphism.

(5.4) Proposition. Let R be a Euclidean ring, then

$$\tilde{H}_{n-2}(\phi(n,R)) : \tilde{H}_{n-2}(\mathcal{O}(n,R)_{\leq n-2}) \to \tilde{H}_{n-2}(T(n,R))$$

is surjective for $n \geq 1$.

Proof. For $n = 1$, both $\mathcal{O}(n,R)_{\leq n-2}$ and $T(n,R)$ are empty, so there is nothing to prove. So take $n \geq 2$. Then for $V \in T(n,R)$

$$\phi(n,R)/V = \mathcal{O}(\dim V,R)$$

is $\dim V-1 = \mathrm{ht}_{T(n,R)}(V)$ spherical by (4.2), and by (5.3)

$$\mathrm{Link}^{+}_{T(n,R)}(V) \cong T(n-\dim V,R)$$

which is by (5.2) $n-\dim V - 2 = (n-2)-\mathrm{ht}_{T(n,R)}(V)-1$-spherical. As $T(n,R)$ is $n-2$-spherical, and $\phi(n,R)$ is a homogeneous morphism, II(3.3) shows that $\tilde{H}_{n-2}(\phi(n,R))$ is surjective.

Now we are ready to give the main result of this section.

(5.5) Theorem. Let R be a field K or a subring of $\mathbb{Q}$, then $\mathcal{A}(n,R)$ is n-spherical for all n.

Proof. Induction on n. The case $n = 0$ being trivial, assume we have for some $n \geq 1$ shown that $\mathcal{A}(k,R)$ is k-spherical for $k < n$.

To show $\mathcal{A}(n,R)$ is n-spherical, we want to use II (4.4)(ii). This means we have to prove $\mathcal{A}(n,R)_{\leq n-1}$ is n-1-spherical and

$$\mathrm{im}(\tilde{H}_{n-1}(\mathcal{A}(n,R)_{\leq n-1}) \to \tilde{H}_{n-1}(\mathcal{A}(n,R)) = 0.$$

First we define

$$\psi = \psi(n,R) : \mathcal{A}(n,R)_{\leq n-1} \to A(n,R)$$

by $\psi(n,R)(v_0,\ldots,v_k) = A(v_0,\ldots,v_k)$. Evidently $\psi(n,R)$ is a morphism of posets such that

$$\mathrm{ht}_{A(n,R)}(\psi(n,R)(v_0,\ldots,v_k)) = \mathrm{ht}_{\mathcal{A}(n,R)_{\leq n-1}}(v_0,\ldots,v_k)$$

Furthermore, for an arbitrary $a+V \in A(n,R)$ we have

$$\psi(n,R)/(a+V) \cong \mathcal{A}(\dim V,R)$$

which is homogeneous of dimension $\dim V = \mathrm{ht}_{A(n,R)}(a+V)$ and even dim V-spherical by the induction hypothesis. Hence $\psi(n,R)$ is a homogeneous morphism. Furthermore $A(n,R)$ is n-1-spherical and because of (5.3)(ii)

$$\mathrm{Link}^+_{a+V}(A(n,R)) \cong T(n - \dim V, R)$$

which is n-dim V - 2 = n-1-$ht_{A(n,R)}(a+V)$-1-spherical by (5.2). So we can apply II (3.3). We conclude that $\mathcal{A}(n,R)_{\leq n-1}$ is n-1-spherical and there is a filtration

$$0 = F_n \subseteq F_{n-1} \subseteq \ldots \subseteq F_0 \subseteq F_{-1} = \tilde{H}_{n-1}(\mathcal{A}(n,R)_{\leq n-1})$$

such that

$$F_{-1}/F_0 \cong \tilde{H}_{n-1}(A(n,R))$$

$$F_q/F_{q+1} \cong \bigoplus_{\substack{a+V\in A(n,R)\\ \dim V = q}} \tilde{H}_q(\psi(n,R)/a+V) \otimes \tilde{H}_{n-q-2}(\mathrm{Link}^+_{A(n,R)}(a+V))$$

(5.6) Claim. For q = n-1,...,0, if we have that $\mathrm{im}(F_{q+1} \to \tilde{H}_{n-1}(\mathcal{A}(n,R)) = 0$ then also $\mathrm{im}(F_q \to \tilde{H}_{n-1}(\mathcal{A}(n,R)) = 0$.

Since $F_n = 0$, this claim shows that $\mathrm{im}(F_0 \to \tilde{H}_{n-1}(\mathcal{A}(n,R))=0$. We first prove this claim, and then we shall show that it follows from this that $\mathrm{im}(F_{-1} \to \tilde{H}_{n-1}(\mathcal{A}(n,R))=0$ i.e. $\mathrm{im}(\tilde{H}_{n-1}(\mathcal{A}(n,R)_{\leq n-1}) \to \tilde{H}_{n-1}(\mathcal{A}(n,R))) = 0$.

Proof of claim (5.6). To prove this, we look more closely at theorems II(2.4) and II(3.3). They give a factorisation

$$\mathcal{A}(n,R)_{\leq n-1} = U_n \xrightarrow{f_n} U_{n-1} \to \ldots \to U_1 \xrightarrow{f_1} U_0 = A(n,R)$$

with

$$U_q = \mathcal{A}(n,R)_{\leq q-1} \amalg A(n,R)_{\geq q}.$$

such that U_q is n-1-spherical and for $q \geq 0$

$$F_q = \mathrm{Ker}(\tilde{H}_{n-1}(\mathcal{A}(n,R)_{\leq n-1}) \to \tilde{H}_{n-1}(U_q))$$

and, identifying $\tilde{H}_{n-1}(U_q)$ with F_{-1}/F_q, we have

$$\mathrm{Ker}(\tilde H_{n-1}(U_{q+1}) \to \tilde H_{n-1}(U_q)) = F_q/F_{q+1}.$$

Now for $a+V \in A(n,R)$ with $\dim V = q$ we have an inclusion morphism

$$\psi/a+V * \mathrm{Link}^+_{A(n,R)}(a+V) \to U_{q+1}$$

By virtue of II(2.4) and II(3.3), this yields an injection

$$\bigoplus_{\substack{a+V\in A(n,R)\\ \dim V = q}} \tilde H_q(\psi/a+V) \otimes \tilde H_{n-2-q}(\mathrm{Link}^+_{A(n,R)}(a+V)) \to \tilde H_{n-1}(U_{q+1})$$

having as its image F_q/F_{q+1}. Now, to prove our claim (5.6) it is enough if for $a+V$ with $\dim V = q$, we construct a subposet $S(V)$ of $\mathcal{A}(n,R)_{\leqslant n-1}$ with the following properties:

a) $S(V)$ is $n-1$-spherical.

b) $\tilde H_{n-1}(S(V)) \to \tilde H_{n-1}(U_{q+1})$ surjects onto the part $\tilde H_q(\psi/a+V) \otimes \tilde H_{n-q-2}(\mathrm{Link}^+_{A(n,R)}(a+V))$.

c) $\mathrm{Im}(\tilde H_{n-1}(S(V)) \to \tilde H_{n-1}(\mathcal{A}(n,R)) = 0$.

First observe we may assume $a = 0$, $V = \langle e_1,\dots,e_q \rangle$; take $W = \langle e_{q+1},\dots,e_n \rangle$. Identify $\psi(n,R)/V$ with $\mathcal{A}(q,R)$ and $(\psi(n,R)/W) \cap \mathcal{O}(n,R)$ with $\mathcal{O}(n-q,R)$. Define

$$S'(V) = H(\mathcal{T}(q,R), \mathcal{O}(n-q,R))$$

cf. II. §1. An element of $S'(V)$ can be described as a sequence $(v_0,\dots,v_k,w_0,\dots,w_l)$ with $(v_0,\dots,v_k) \in \mathcal{K}(q,R)$ $(w_0,\dots,w_l) \in \mathcal{O}(n-q,R)$, with possibly $k = -1$ or $l = -1$ though not both. It is easy to see in this case $(v_0,\dots,v_k,w_0,\dots,w_l) \in \mathcal{K}(n,R)$ and we have an inclusion

$$\eta'_V : S'(V) \to \mathcal{A}(n,R).$$

We define

$$S(V) = H(\mathcal{A}(q,R), \mathcal{O}(n-q,R)_{\leqslant n-q-2}).$$

Then $S(V)$ is a subposet of $S'(V)$ and we can define $\eta_V = \eta'_V|S(V)$. In fact η_V lands in $\mathcal{T}(n,R)_{\leqslant n-1}$.

By (4.2) $\mathcal{O}(n-q,R)$ is $n-q-1$-spherical, so by II(4.4)(i) $\mathcal{O}(n-q,R)_{\leqslant n-q-2}$ is $n-q-2$-spherical. By II(1.2)(iv), I(1.5) and the induction hypothesis we conclude $S'(V)$ is n-spherical, $S(V)$ is $n-1$-spherical, hence a). We have a commutative diagram

$$\begin{array}{ccc} S(V) & \xrightarrow{\eta_V} & \mathcal{A}(n,R)_{\leqslant n-1} \\ \downarrow & & \downarrow \\ S'(V) & \xrightarrow{\eta'_V} & \mathcal{A}(n,R) \end{array}$$

As $\tilde{H}_{n-1}(S'(V)) = 0$, $\mathrm{im}(\tilde{H}_{n-1}(S(V)) \to \tilde{H}_{n-1}(\mathcal{A}(n,R))) = 0$, i.e. c).

To settle b) we have to look more closely at $f_{q+2}\ldots f_n\eta_V : S(V) \to U_{q+1}$. We have

$$f_{q+2}\ldots f_n\eta_V(v_0,\ldots,v_k,w_0,\ldots,w_l) = \begin{cases} (v_0,\ldots,v_k,w_0,\ldots,w_l) & k+l+1 \leqslant q \\ A(v_0,\ldots,v_k,w_0,\ldots,w_l) & k+l+1 > q \end{cases}$$

We define a $\gamma : S(V) \to U_{q+1}$ by

$$\gamma(v_0,\ldots,v_k,w_0,\ldots,w_l) = \begin{cases} (v_0,\ldots,v_k) & l = -1 \\ V \oplus \langle w_0,\ldots,w_l\rangle & l \geqslant 0. \end{cases}$$

Evidently $f_{q+2}\ldots f_n\eta_V(v_0,\ldots,v_k,w_0,\ldots,w_l) \leqslant \gamma(v_0,\ldots,v_k,w_0,\ldots,w_l)$ so by I(2.1)(ii) $|f_{q+2}\ldots f_n\eta_V|$ and $|\gamma|$ are homotopic, so to prove b) we have to show $\tilde{H}_{n-1}(\gamma)$ maps $\tilde{H}_{n-1}(S(V))$ onto $\tilde{H}_q(\mathcal{A}(q,R)) \otimes \tilde{H}_{n-q-2}(\mathrm{Link}^+_{A(n,R)}(V))$.

Above we saw that $\tilde{H}_q(\mathcal{A}(q,R)) \otimes \tilde{H}_{n-q-2}(\mathrm{Link}^+_{A(n,R)}(V))$ in $\tilde{H}_{n-1}(U_{q+1})$ comes from the inclusion

$$\mathcal{A}(q,R) * \mathrm{Link}^+_{A(n,R)}(V) \to U_{q+1}\,.$$

Now γ maps $S(V)$ onto $\mathcal{A}(q,R) * \mathrm{Link}^+_{A(n,R)}(V)$. Identifying $\mathrm{Link}^+_{A(n,R)}(V)$ with $T(n-q,R)$ by means of (5.3), and defining

$$\gamma_0 : \mathcal{A}(q,R) * \mathcal{O}(n-q,R)_{\leqslant n-q-2} \to \mathcal{A}(q,R) * T(n-q,R)$$

by $\gamma_0|\mathcal{A}(q,R) = \mathrm{id}_{\mathcal{A}(q,R)}$, $\gamma_0|\mathcal{O}(n-q,R)_{\leqslant n-q-2} = \phi(n-q,R)$ we see we can factorise γ as

$$\gamma = \gamma_0 \circ h(\mathcal{A}(q,R), \mathcal{O}(n-q,R)_{\leqslant n-q-2}).$$

From II (1.2)(iii) we see $|h(\mathcal{A}(q,R),\mathcal{O}(n-q,R)_{\leq n-q-2})|$ is a homotopy equivalence. So it remains to show $\tilde{H}_{n-1}(\gamma_0)$ is surjective.

By the induction hypothesis, $\mathcal{A}(q,R)$ is q-spherical, $T(n-q,R)$ is n-q-2-spherical by (5.2), $\mathcal{O}(n-q,R)_{\leq n-q-2}$ is n-q-2-spherical, and by II (1.2)

$$|\mathcal{A}(q,R) * \mathcal{O}(n-q,R)_{\leq n-q-2}| \cong \mathrm{Hopf}(|\mathcal{A}(q,R)|,|\mathcal{O}(n-q,R)_{\leq n-q-2}|)$$

$$|\mathcal{A}(q,R) * T(n-q,R)| \cong \mathrm{Hopf}(|\mathcal{A}(q,R)|,|T(n-q,R)|)$$

I (1.5) tells us both spaces are n-1-spherical. A glance at the proof of I (1.5) shows that the map

$$\tilde{H}_{n-1}(\gamma_0) : \tilde{H}_q(\mathcal{A}(q,R)) \otimes \tilde{H}_{n-q-2}(\mathcal{O}(n-q,R)_{\leq n-q-2}) \to$$
$$\to \tilde{H}_q(\mathcal{A}(q,R)) \otimes \tilde{H}_{n-q-2}(T(n-q,R))$$

is equal to $\tilde{H}_q(id) \otimes \tilde{H}_{n-q-2}(\phi(n-q,R))$. By (5.4), the map $\tilde{H}_{n-q-2}(\phi(n-q,R))$ is surjective. This finishes the proof of claim (5.6).

We have shown so far that $\mathrm{im}(F_0 \to \tilde{H}_{n-1}(\mathcal{A}(n,R))) = 0$. So we have a factorisation

$$\begin{array}{ccc} F_{-1} = \tilde{H}_{n-1}(\mathcal{A}(n,R)_{\leq n-1}) & \searrow & \\ \tilde{H}_{n-1}(\psi(n,R)) \downarrow & & \tilde{H}_{n-1}(\mathcal{A}(n,R)) \\ F_{-1}/F_0 = \tilde{H}_{n-1}(A(n,R)) & \nearrow & \end{array}$$

Denote the boundary $\mathcal{C}_n(\mathcal{A}(n,R)) \to \mathcal{C}_{n-1}(\mathcal{A}(n,R))$ by δ. We shall show: given a generator $g(v_0,\ldots,v_k)$ of $\tilde{H}_{n-1}(A(n,R))$, which we represent by a cycle as in (5.2), there is a $k \in \mathcal{C}_n(\mathcal{A}(n,R))$ such that $\delta k \in \mathcal{C}_{n-1}(\mathcal{A}(n,R)_{\leq n-1})$ and δk is mapped by $\psi(n,R)$ to $g(v_0,\ldots,v_n)$. This proves $\mathrm{im}(F_{-1} \to \tilde{H}_{n-1}(\mathcal{A}(n,R))) = 0$.

We have three cases.

(i) R is a field K. In this case $(v_0,\ldots,v_n)$ in general position in K^n means $(v_0,\dots,v_n) \in \mathcal{A}(n,K)$. We can identify $\mathcal{A}(n,K)/(v_0,\dots,v_n)$ with $\mathcal{O}(N_*\Delta_n)$ by sending $i \mapsto v_{i-1}$ for $i = 1,\dots,n+1$. Take $k = (-1)^n\Delta(1,\dots,n+1)$. Then $\delta k = (-1)^n\gamma(1,\dots,n+1) \in \mathcal{C}_{n-1}(\mathcal{A}(n,R)_{\leqslant n-1})$. From our definitions it is clear that $\psi(n,K)$ maps δk to $g(v_0,\dots,v_n)$.

(ii) $R = \mathbb{Z}$. Here also, if $(w_0,\dots,w_n) \in \mathcal{A}(n,\mathbb{Z})$ then $g(w_0,\dots,w_n)$ goes to zero in $\widetilde{H}_{n-1}(\mathcal{A}(n,\mathbb{Z}))$ by the same reasoning. We prove that each $g(v_0,\dots,v_n)$ can be expressed as a sum of $g(w_0,\dots,w_n)$ with $(w_0,\dots,w_n) \in \mathcal{A}(n,\mathbb{Z})$.

From the remark at the end of the proof of (5.2) we find for $v_0,\dots,v_n$, $w \in \mathbb{Z}^n$

$$g(v_0,\dots,v_n) = \sum_{i=0}^{n} g(v_0,\dots,v_{i-1},w,v_{i+1},\dots,v_n) .$$

The problem is to choose w suitably.

Define for $v_0,\dots,v_n \in \mathbb{Z}^n$

$$|(v_0,\dots,v_n)| = |\det(v_1-v_0,\dots,v_n-v_0)|.$$

If $|(v_0,\dots,v_n)| = 0$, then $g(v_0,\dots,v_n) = 0$ and if $|(v_0,\dots,v_n)| = 1$ then $g(v_0,\dots,v_n)$ goes to zero in $\widetilde{H}_{n-1}(\mathcal{A}(n,\mathbb{Z}))$ as $(v_0,\dots,v_n) \in \mathcal{A}(n,\mathbb{Z})$ in this case. So assume $|(v_0,\dots,v_n)| = t > 1$. We want to find a $w \in \mathbb{Z}^n$ such that for $i = 0,\dots,n$

$$|(v_0,\dots,v_{i-1},w,v_{i+1},\dots,v_n)| < t$$

Induction then shows $g(v_0,\dots,v_n)$ goes to zero in $\widetilde{H}_{n-1}(\mathcal{A}(n,\mathbb{Z}))$ for all $v_0,\dots,v_n$, using the formula above.

Now take v_0 as origin. Then $v_1,\dots,v_n$ is a basis of $\mathbb{Q}^n$ over $\mathbb{Q}$, so we can write $w = \sum_{i=1}^{n} \alpha_i v_i$ with $\alpha_i \in \mathbb{Q}$. We first prove there is a $w \in \mathbb{Z}^n$ with the properties $|\alpha_i| < 1$ for all i, and $|\sum_{i=1}^{n} \alpha_i - 1| < 1$. We finish by showing this is a good choice for w.

Define

$$P(v_1,\dots,v_n) = \{\sum_{i=1}^{n} \lambda_i v_i \mid \lambda_i \in \mathbb{R}, 0 \leqslant \lambda_i < 1\},$$

then the map

$$P(v_1,\dots,v_n) \to \mathbb{R}^n/(\mathbb{Z}v_1 + \dots + \mathbb{Z}v_n)$$

is a bijection. As

$$\mathbb{Z}^n/(\mathbb{Z}v_1 + \dots + \mathbb{Z}v_n) \hookrightarrow \mathbb{R}^n/(\mathbb{Z}v_1 + \dots + \mathbb{Z}v_n)$$

and

$$|\mathbb{Z}^n/(\mathbb{Z}v_1 + \dots + \mathbb{Z}v_n)| = |\det(v_1,\dots,v_n)| > 1$$

we can take a non-zero element in $\mathbb{Z}^n/(\mathbb{Z}v_1 + \dots + \mathbb{Z}v_n)$ which yields a $w_0 \in \mathbb{Z}^n \cap (P(v_1,\dots,v_n)\backslash\{v_0\})$. Write $w_0 = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \in \mathbb{Q}$, $0 \leqslant \lambda_i < 1$. We have $\lambda_i \neq 0$ for at least one i, and we have two cases

a) $0 < \sum_{i=1}^{n} \lambda_i < 2$. In this case we take $\alpha_i = \lambda_i$ for all i, i.e. $w = w_0$.

b) $\sum_{i=1}^{n} \lambda_i \geqslant 2$. Let $J = \{i \mid \lambda_i \neq 0\}$. Then $J \neq \phi$ and we have $\sum_{i=1}^{n} \lambda_i = \sum_{i\in J} \lambda_i$. Now take $d = [\sum_{i=1}^{n} \lambda_i] - 1$, then $1 \leqslant d < |J|$. Now select distinct elements $i_1,\dots,i_d \in J$ and define $\alpha_i = \lambda_i - 1$ for $i \in \{i_1,\dots,i_d\}$ and $\alpha_i = \lambda_i$ if $i \notin \{i_1,\dots,i_d\}$. Then obviously $|\alpha_i| < 1$. As $d+1 \leqslant \sum_{i=1}^{n} \lambda_i < d+2$ and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \lambda_i - d$ we find $1 \leqslant \sum_{i=1}^{n} \alpha_i < 2$. So take $w = \sum_{i=1}^{n} \alpha_i v_i = w_0 - v_{i_1} - \dots - v_{i_d}$. Then $w \in \mathbb{Z}^n$, and it has the desired properties.

Now, for $i = 1,\dots,n$ we have

$$\begin{aligned}|(v_0,\dots,v_{i-1},w,v_{i+1},\dots,v_n)| &= |\det(v_1,\dots,v_{i-1},\sum_{j=1}^{n}\alpha_j v_j, v_{i+1},\dots,v_n)| \\ &= |\alpha_i||\det(v_1,\dots,v_n)| \\ &< |(v_0,v_1,\dots,v_n)|\end{aligned}$$

because $|\alpha_i| < 1$. Furthermore, we find

$$\begin{aligned} |(w,v_1,\dots,v_n)| &= |\det(v_1-w,\dots,v_n-w)| \\ &= |\det(\sum_{i=1}^{n}\alpha_i v_i - v_1, v_2-v_1,\dots,v_n-v_1)| \\ &= |\det((\sum_{i=1}^{n}\alpha_i - 1)v_1, v_2-v_1,\dots,v_n-v_1)| \\ &= |\sum_{i=1}^{n}\alpha_i - 1|\,|\det(v_1,\dots,v_n)| \\ &< |(v_0,v_1,\dots,v_n)| \end{aligned}$$

since $0 < \sum_{i=1}^{n}\alpha_i < 2$. This finishes our proof of case ii).

iii) R is a subring of $\mathbb{Q}$. This case is essentially a modification of the previous one. Write $R = \mathbb{Z}[\frac{1}{p} | p \in S]$. Observe first that for given $v_1,\dots,v_n \in R^n$

$$|R^n/(Rv_1 + \dots + Rv_n)| = |\det(v_1,\dots,v_n)|_S .$$

This is best seen by taking $\alpha_1,\dots,\alpha_n \in R^*$ such that $\alpha_1 v_1,\dots,\alpha_n v_n \in \mathbb{Z}^n$ and noticing that

$$R^n/(Rv_1+\dots+Rv_n) \cong (\mathbb{Z}^n/(\mathbb{Z}(\alpha_1 v_1)+\dots+\mathbb{Z}(\alpha_n v_n))) \otimes_{\mathbb{Z}} R$$

from which it follows easily.

Now proceed as in case ii). Define for $v_0,\dots,v_n \in R^n$

$$|(v_0,\dots,v_n)|_S = |\det(v_1-v_0,\dots,v_n-v_0)|_S .$$

We have to show: given $v_0,\dots,v_n \in R^n$ with $|(v_0,\dots,v_n|_S = t > 1$ there is a $w \in R^n$ such that for $i = 0,\dots,n$

$$|(v_0,\dots,v_{i-1},w,v_{i+1},\dots,v_n)|_S < t.$$

Take w_0 representing a nonzero element of $R^n/(Rv_1+\dots+Rv_n)$. Write $w_0 = \sum_{i=1}^{n}\lambda_i v_i$. Then $\lambda_i = \lambda_{i,S}\lambda'_{i,S}$ and there is an $m \in \mathbb{Z}$ with $|m|_S = 1$ such that $m\lambda_{i,S} \in \mathbb{Z}$ for all i. Now $m \in R^*$, so mw_0 also represents a nonzero element in $R^n/(Rv_1+\dots+Rv_n)$. Hence we may assume $w_0 = \sum_{i=1}^{n}\lambda_i v_i$ with $\lambda_{i,S} \in \mathbb{Z}$ for $i = 1,\dots,n$.

Denote $R_C = \mathbb{Z}[\frac{1}{p} | p \notin S]$. Then for $r \in R_C$ we have $r_S \in \mathbb{Z}$ so because $|r| = |r|_S|r_S|$ we have $|r|_S \leqslant |r|$. We have $w_0 = \sum_{i=1}^{n}\lambda_i v_i$

with $\lambda_i \in R_C$. Now, by adding ones to the λ_i, we find a w still representing a nonzero element in $R^n/(Rv_1+\ldots+Rv_n)$ with $w = \Sigma\alpha_i v_i$, such that $\alpha_i \in R_C$ and $|\alpha_i| < 1$ for $i = 1,\ldots,n$, and $|\Sigma\alpha_i - 1| < 1$. As $\alpha_i \in R_C$, also $|\alpha_i|_S < 1$ and as R_C is a ring, $\Sigma\alpha_i - 1 \in R_C$ so $|\Sigma\alpha_i-1|_S \leqslant |\Sigma\alpha_i-1| < 1$.

We finish the proof by showing as in case ii) that this w satisfies our requirements, now using the $|.|_S$-norm.

For $(v_0,\ldots,v_p) \in \mathcal{A}(n,R)$ with $p < n$, we have $\mathcal{A}(n,R)_{(v_0,\ldots,v_p)} = \mathcal{O}(n,p,R)$ so combining II(5.4), (4.2) and (5.5) yields

(5.7) <u>Corollary</u>. Let R be a field or a subring of $\mathbb{Q}$. Then the poset $\mathcal{A}(n,R)$ is n-dim CM for all n.

We finish this section by introducing an affine analogue of $\mathcal{O}(n,k,R)$, and showing this is CM for a field or a subring of $\mathbb{Q}$.

(5.8) <u>Definition</u>. Let R be a ring, $0 \leqslant k < n$. Then we define

$$\mathcal{A}(n,k,R) = \{(v_0,\ldots,v_p) \in \mathcal{O}(R^n) \mid (v_1-v_0,\ldots,v_p-v_0) \in \mathcal{O}(n,k,R)\}$$

$$\mathcal{A}(n,n,R) = \{(v) \in \mathcal{O}(R^n) \mid v \text{ arbitrary}\}.$$

It is clear that $\mathcal{A}(n,k,R)$ is full of dimension n-k for all n and k with $0 \leqslant k \leqslant n$. One easily sees that $\mathcal{A}(n,k,R) \cong \mathcal{A}(n-k,R)\langle R^k\rangle$. So by II(6.3) and (5.7) we have

(5.9) <u>Corollary</u>. If R is a field or a subring of $\mathbb{Q}$ then $\mathcal{A}(n,k,R)$ is n-k-dim CM for $0 \leqslant k \leqslant n$.

6. General acyclicity theorems.

In this section, we consider a ring R with Jacobson radical J. We want to compare $\mathcal{O}(n,R)$ resp. $\mathcal{A}(n,k,R)$ with $\mathcal{O}(n,R/J)$ resp. $\mathcal{A}(n,k,R/J)$. We have the following result.

(6.1) Proposition. Let R be a ring, J an ideal contained in its Jacobson radical. Then

(i) $\mathcal{O}(n,R) \cong \mathcal{O}(n,R/J)< J^n >$

(ii) $\mathcal{A}(n,k,R) \cong \mathcal{A}(n,k,R/J)< J^n >$

Proof. (i) Denote the projection $R \to R/J$ by κ. This gives a projection $\kappa : R^n \to (R/J)^n$.

Take a sequence $v_0,\ldots,v_p$ of elements of R^n such that $(\kappa v_0,\ldots,\kappa v_p) \in \mathcal{O}(n,R/J)$. Then we can find $v_{p+1},\ldots,v_{n-1} \in R^n$ such that $(\kappa v_0,\ldots,\kappa v_p,\kappa v_{p+1},\ldots,\kappa v_{n-1}) \in \mathcal{O}(n,R/J)$. But then

$$\det(\kappa v_0,\ldots,\kappa v_{n-1}) = \kappa \det(v_0,\ldots,v_{n-1}) \in (R/J)^*$$

which since J is contained in the Jacobson radical of R, implies

$$\det(v_0,\ldots,v_{n-1}) \in R^*$$

i.e. $(v_0,\ldots,v_{n-1}) \in \mathcal{O}(n,R)$ whence $(v_0,\ldots,v_p) \in \mathcal{O}(n,R)$.

Now, let $\sigma : R/J \to R$ be a section of κ. It gives a section $\sigma : (R/J)^n \to R^n$ of $\kappa : R^n \to (R/J)^n$. Define

$$\psi : \mathcal{O}(n,R/J)< J^n > \to \mathcal{O}(n,R)$$

for $(v_0,\ldots,v_p) \in \mathcal{O}(n,R/J)$, $s_0,\ldots,s_p \in J^n$ by

$$\psi((v_0,s_0),\ldots,(v_p,s_p)) = (\sigma v_0+s_0,\ldots,\sigma v_p+s_p)$$

As $\kappa(\sigma v_i+s_i) = v_i$, we find that $\psi((v_0,s_0),\ldots,(v_p,s_p)) \in \mathcal{O}(n,R)$. It is left to the reader to verify that ψ is an isomorphism of posets.

(ii) This is analogous to (i).

Combining II(6.3), (4.5), (5.9) and (6.1) yields our main acyclicity theorem.

(6.2) Theorem. Let R be a ring, J an ideal contained in its Jacobson radical.

(i) If R/J is Euclidean, then $\mathcal{O}(n,R)$ is n-1-dim CM for all n.

(ii) If R is local or R/J is a subring of $\mathbb{Q}$, then $\mathcal{H}(n,k,R)$ is n-k-dim CM for $0 \leq k \leq n$.

Remark. $\mathbb{F}_2[X]$ is Euclidean, but $\mathcal{H}(1,0,\mathbb{F}_2[X])$ is not connected and hence not CM, as one easily sees.

IV Posets and group homology.

1. Group actions on posets.

To compare the homology of a group G with the homology of a certain subgroup G_0, we construct a topological space X such that $H_iG = H_iX$ and $H_iX = H_iG_0$ for i within a certain range. In the first two sections of this chapter we describe a general setting to solve this kind of problem. In section 3 and 4 we apply the theory to compare the homology of $GL_n(R)$ and $GL_{n+1}(R)$ for certain rings R.

Let $F \subseteq \mathcal{O}(V)$ be full of dimension d. Suppose we have a group G acting on V. Then we let G act on $\mathcal{O}(V)$ by defining for $g \in G$ and $(v_0,\dots,v_p) \in \mathcal{O}(V)$ that $g(v_0,\dots,v_p) = (gv_0,\dots,gv_p)$. If moreover $GF \subseteq F$ then we say G acts on F. We say G acts transitively on F if we have for each pair of elements of maximal height $(v_0,\dots,v_d)$, $(w_0,\dots,w_d) \in F$ a $g \in G$ such that $g(v_0,\dots,v_d) = (w_0,\dots,w_d)$. It follows that if for any p we have two elements $(v_0,\dots,v_p),(w_0,\dots,w_p) \in F$ then there is a $g \in G$ such that $g(v_0,\dots,v_p) = (w_0,\dots,w_p)$.

<u>Examples</u>. (Compare II §4).

1) Let $F = \mathcal{O}(n)$, $G = S_n$; in this case S_n acts transitively on $\mathcal{O}(n)$.

2) For any ring R, the group $GL_n(R)$ acts transitively on $\mathcal{O}(n,R)$.

3) Let R be a ring and define $GA_n(R) = GL_n(R) \ltimes R^n$ the semi-direct product of $GL_n(R)$ and R^n. We call $GA_n(R)$ the general affine group. It acts transitively on $\mathcal{A}(n,R)$, by $(g,a)(v)=(gv+a)$.

In the remainder of this section, G is a group acting transitively on a full subposet $F \subseteq \mathcal{O}(V)$ of dimension d. A is a commutative ring. We define a category $\mathcal{K}(F,G)$ as follows:

$$\mathrm{Obj}\ \mathcal{K}(F,G) = F$$

and the set of morphisms between $\vec{v},\vec{w} \in F$ is

$$\mathrm{Mor}_{\mathcal{K}(F,G)}(\vec{v},\vec{w}) = \{g \in G \mid g\vec{v} \leqslant \vec{w}\}$$

i.e. we have $g : \vec{v} \to \vec{w}$ if and only if $g\vec{v} \leqslant \vec{w}$. To prove $\mathcal{K}(F,G)$ is an honest category, we have to show: if $g : \vec{v} \to \vec{w}$, $g' : \vec{u} \to \vec{v}$, then $gg' : \vec{u} \to \vec{w}$. Well, we have $g\vec{v} \leqslant \vec{w}$ and $g'\vec{u} \leqslant \vec{v}$ so $gg'\vec{u} \leqslant g\vec{v} \leqslant \vec{w}$.

We can construct a functor

$$\pi = \pi(F,G) : \mathcal{K}(F,G) \to G$$

by sending $\vec{v} \mapsto *$ on objects. For $g : \vec{v} \to \vec{w}$ we define of course $\pi(g) = g$. We want to show π is a cofibred functor and compute the cobase-change g_* for $g \in G$.

We have $\pi/* = \{(\vec{v},g) \mid \vec{v} \in F,\ g \in G\}$ and $\pi^{-1}(*) = F$. The inclusion $i : F = \pi^{-1}(*) \to \pi/*$ sends $\vec{v} \mapsto (\vec{v},e)$. The functor $k : \pi/* \to \pi^{-1}(*)$ given by $(\vec{v},g) \mapsto g\vec{v}$ is seen to be left adjoint to i. We compute for $g \in G$ the cobase change g_* as

$$\vec{v} \mapsto (\vec{v},g) \mapsto g\vec{v}$$

and indeed $(gh)_* = g_*h_*$ for $g,h \in G$.

Theorem I (3.3) then yields a spectral sequence

$$E^2_{pq} = H_p(G,H_q(F,A)) \Rightarrow H_{p+q}(\mathcal{K}(F,G),A)$$

If F is d-spherical with $d \geqslant 1$, then by the universal coefficient theorem $H_d(F,A)$ is free over A, $H_0(F,A) = A$, and the spectral sequence degenerates to a long exact Gysin sequence

$$\ldots \to H_{i+1}(G,A) \to H_{i-d}(G,H_d(F,A)) \to H_i(\mathcal{K}(F,G),A) \to H_i(G,A) \to \ldots$$

see Spanier [13] Chap.9, sec. 3, theorem 3.

(1.1) Proposition. Let $F \subseteq \mathcal{O}(V)$ be full of dimension d, and G act transitively on F. If F is d-spherical then we have for all commutative rings A

$$H_i(\pi) : H_i(\ast(F,G),A) \xrightarrow{\sim} H_i(G,A) \qquad i < d$$

$$H_d(\pi) : H_d(\ast(F,G),A) \twoheadrightarrow H_d(G,A)$$

Proof. If d = 0 this is obvious. If d > 0, use the exact Gysin sequence above. By I §3, we can identify the map $H_i(\ast(F,G),A) \to H_i(G,A)$ in the Gysin sequence with $H_i(\pi)$.

In the next section, we shall compute $H_*(\ast(F,G),A)$ in terms of the homology of certain subgroups of G. Though the description seems fairly complicated, it yields in our applications that $H_i(G_0,A) \cong H_i(\ast(F,G),A)$ for a certain subgroup G_0 of G and small i.

2. The fundamental spectral sequence.

In our description of the homology of $\ast(F,G)$ an important rôle will be played by the category Q(n), which we define for $n \geq 1$ as follows:

$$\text{Obj } Q(n) = \{\{1,\ldots,p\} \mid p \leq n\}$$

a morphism $\phi : \{1,\ldots,p\} \to \{1,\ldots,q\}$ is a map such that $\phi(i) < \phi(j)$ for $i < j$. We want to give a simple way to calculate $H_*(Q(n)^0,\mathcal{L})$ for any system of coefficients on the opposite category $Q(n)^0$.

Define for $0 \leq i \leq k \leq n$, the morphisms

$$\delta_i^k : \{1,\ldots,k\} \to \{1,\ldots,k+1\}$$

by $\delta_i^k(l) = l$ if $l \leq i$, $\delta_i^k(l) = l+1$ if $l > i$. For a system of coefficients $\mathcal{L}$ on $Q(n)^0$ we define the complex $\mathcal{C}^{red}(Q(n)^0,\mathcal{L})$ by

$$\mathcal{C}^{red}_{k-1}(Q(n)^0,\mathcal{L}) = \mathcal{L}(\{1,\ldots,k\})$$

and

$$\delta_k : \mathcal{L}(\{1,\ldots,k+1\}) \to \mathcal{L}(\{1,\ldots,k\})$$
$$\delta_k = \sum_{i=0}^{k} (-1)^i \mathcal{L}(\delta^k_i)$$

Then by a brute force computation $\delta_k\ \delta_{k+1} = 0$ and in fact we have

(2.1) Proposition. For any system of coefficients $\mathcal{L}$ on $Q(n)^0$ we have

$$H_*(Q(n)^0,\mathcal{L}) \cong H_*(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{L})).$$

Proof. It is easy to see that $\mathcal{L} \mapsto H_*(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{L}))$ also defines a homology theory on the abelian category of systems of coefficients on $Q(n)^0$. General homological algebra yields that both homology theories are equal if

a) $H_0(Q(n)^0,\mathcal{L}) \cong H_0(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{L}))$.

b) For any $\mathcal{L}$ there is an acyclic covering $\mathcal{P}$ of $\mathcal{L}$ such that also $H_i(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{P}) = 0$ for $i > 0$.

To prove a) observe that

$$\begin{aligned} H_0(Q(n)^0,\mathcal{L}) &= \varinjlim_{X\in Q(n)^0} \mathcal{L}(X) \\ &= \mathcal{L}\{1\}/[(\mathcal{L}(\delta^1_0) - \mathcal{L}(\delta^1_1))\mathcal{L}(\{1,2\})] \\ &= H_0(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{L})). \end{aligned}$$

For b) take $\mathcal{P}$ to be the acyclic covering of $\mathcal{L}$ as constructed in the proof of proposition I (3.2). To prove $H_i(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{P})) = 0$ for $i > 0$, we only have to show for $\{1,\ldots,k\} \in Q(n)$ and L an abelian group that for $i > 0$

$$H_i(\mathcal{C}^{red}_*(Q(n)^0,\mathcal{P}(\{1,\ldots,k\},L))) = 0$$

We have

$$\mathcal{C}^{red}_{j-1}(Q(n)^0,\mathcal{P}(\{1,\ldots,k\},L)) = \mathcal{P}(\{1,\ldots,k\},L)(\{1,\ldots,j\})$$

$$= \coprod_{\phi:\{1,\dots,j\}\to\{1,\dots,k\}} L$$

$$= \mathcal{C}_{j-1}(\Delta^{k-1},L)$$

Comparing bounderies of $\mathcal{C}_*^{red}(Q(n)^o,\mathcal{P}(\{1,\dots,k\},L))$ and $\mathcal{C}_*(\Delta^{k-1},L)$ one sees they are equal, so we find

$$H_i(\mathcal{C}_*^{red}(Q(n)^o,\mathcal{P}(\{1,\dots,k\},L))) = H_i(\Delta^{k-1},L)$$

and $H_i(\Delta^{k-1},L) = 0$ for $i > 0$.

Assume once more F is a full subposet of $\mathcal{O}(V)$ of dimension d, and G is a group acting transitively on F. We define a functor

$$\rho = \rho(F,G) : \mathcal{K}(F,G) \to Q(d+1)$$

On objects we have

$$\rho(F,G)(v_1,\dots,v_p) = \{1,\dots,p\}.$$

Let $g : (v_1,\dots,v_p) \to (w_1,\dots,w_q)$ be a morphism, then we have $g(v_1,\dots,v_p) \leq (w_1,\dots,w_q)$ so there is a unique map $\phi : \{1,\dots,p\} \to \{1,\dots,q\}$ such that $\phi i < \phi j$ for $i < j$ and $gv_i = w_{\phi i}$ for $i = 1,\dots,p$. We define

$$\rho(g : (v_1,\dots,v_p) \to (w_1,\dots,w_q)) = \phi.$$

It is obvious this indeed defines a functor.

(2.2) Proposition. Let G act transitively on the full F of dimension d. Then $\rho(F,G) : \mathcal{K}(F,G) \to Q(d+1)$ is a fibred functor.

Proof. First we determine $\{1,\dots,k\}\backslash\rho$. We have

$$\{1,\dots,k\}\backslash\rho = \{((w_1,\dots,w_l),\phi) \mid \phi : \{1,\dots,k\} \to \{1,\dots,l\}\}$$

The inclusion functor $i_k : \rho^{-1}\{1,\dots,k\} \to \{1,\dots,k\}\backslash\rho$ is given by $i_k(w_1,\dots,w_k) = ((w_1,\dots,w_k),id)$, with id the identity morphism $\{1,\dots,k\} \to \{1,\dots,k\}$.

We have to exhibit a right adjoint r_k to i_k. This goes as follows: define

$$r_k : \{1,\dots,k\}\backslash\rho \to \rho^{-1}\{1,\dots,k\}$$

by

$$r_k((w_1,\dots,w_l),\phi) = (w_{\phi 1},\dots,w_{\phi k})$$

Now, for $((w_1,\dots,w_l),\phi) \in \{1,\dots,k\}\backslash\rho$ the morphism $e : ((w_{\phi 1},\dots,w_{\phi k}),id) \to ((w_1,\dots,w_l),\phi)$ with $e \in G$ the identity enjoys the following universal property: given a morphism $g : ((v_1,\dots,v_k),id) \to ((w_1,\dots,w_l),\phi)$ there is a unique factorisation

$$\begin{array}{ccc} ((v_1,\dots,v_k),id) & \xrightarrow{\;g\;} & ((w_1,\dots,w_l),\phi) \\ & \searrow^{g} & \downarrow e \\ & & ((w_{\phi 1},\dots,w_{\phi k}),id) \end{array}$$

yielding an isomorphism

$$Mor_{\{1,\dots,k\}\backslash\rho}(i_k(v_1,\dots,v_k),((w_1,\dots,w_l),\phi)) \cong$$
$$\cong Mor_{\rho^{-1}\{1,\dots,k\}}((v_1,\dots,v_k),r_k((w_1,\dots,w_l),\phi))$$

which clearly means r_k is right adjoint to i_k.

We have to compute the base-change: take $\phi : \{1,\dots,k\} \to \{1,\dots,l\}$ to be a morphism in $Q(d+1)$. Then for $(w_1,\dots,w_l) \in \rho^{-1}\{1,\dots,l\}$

$$\phi^*(w_1,\dots,w_l) = r_k((w_1,\dots,w_l),\phi)$$
$$= (w_{\phi 1},\dots,w_{\phi k})$$

and hence $(\phi\psi)^* = \psi^*\phi^*$, whenever the product is defined.

Both the possibility of computing the homology of a local system of coefficients on $Q(d+1)^0$ and the fact that $\rho(F,G)$ is a fibred functor suggests that we must consider the functor

$\rho(F,G)^0 : \mathcal{X}(F,G)^0 \to Q(d+1)^0$ of opposite categories. It is a cofibred functor. We have $\rho(F,G)^0/\{1,\dots,k\} = (\{1,\dots,k\}\backslash\rho(F,G))^0$. As the homology of a category and its opposite are the same for a constant system with value say, L, we find by I(3.3)

(2.3) Theorem. Let F be full of dimension d. Let G act transitively on F, then we have for any abelian group L a spectral sequence

$$E^2_{pq}(F,G)=H_p(Q(d+1)^0,\{1,\dots,k\}\mapsto H_q(\{1,\dots,k\}\backslash\rho(F,G),L)) \Rightarrow$$
$$\Rightarrow H_{p+q}(\mathcal{X}(F,G),L) .$$

As $\rho(F,G)^0$ is cofibred, the system of coefficients $\{1,\dots,k\} \mapsto H_q(\{1,\dots,k\}\backslash\rho(F,G),L)$ coincides with the system of coefficients $\{1,\dots,k\} \mapsto H_q(\rho(F,G)^{-1}\{1,\dots,k\},L)$. We finish this section with a more manageable description of this.

Choose an element of maximal height $(e_1,\dots,e_{d+1}) \in F$. Define for $0 \leqslant i \leqslant d$ the group

$$G_i = \mathrm{Stab}_G(e_1,\dots,e_{i+1})=\{g \in G \mid ge_s=e_s,\ 1 \leqslant s \leqslant i+1\}$$

Since in $\rho^{-1}\{1,\dots,k\}$ each morphism is an isomorphism, the inclusion

$$j_k : G_{k-1} \to \rho^{-1}\{1,\dots,k\}$$

of G_{k-1} in $\rho^{-1}\{1,\dots,k\}$ as group of automorphisms of $(e_1,\dots,e_k)$ is an equivalence of categories. It follows from (2.2) that $H_*(G_{k-1},L) \cong H_*(\{1,\dots,k\}\backslash\rho,L)$.

Let $\phi : \{1,\dots,k\} \to \{1,\dots,l\}$ be a morphism in $Q(d+1)$. We want to describe

$$H_*(\phi^*) : H_*(\{1,\dots,l\}\backslash\rho,L) \to H_*(\{1,\dots,k\}\backslash\rho,L) .$$

Choose an element $g_\phi \in G$ such that

$$g_\phi^{-1}(e_1,\ldots,e_k) = (e_{\phi 1},\ldots,e_{\phi k})$$

For $g \in G_{l-1}$, $s = 1,\ldots,k$ we have $[\mathrm{Int}_G(g_\phi)(g)]e_s = e_s$, so $\mathrm{Int}_G(g_\phi)$ defines a homomorphism $G_{l-1} \to G_{k-1}$. Hence we have a homomorphism

$$H_*(\mathrm{Int}_G(g_\phi)) : H_*(G_{l-1},L) \to H_*(G_{k-1},L)$$

which we claim to be $H_*(\phi^*)$ under our identifications.

Consider the diagram of functors

$$\begin{array}{ccc}
\mathrm{Stab}_G(e_{\phi 1},\ldots,e_{\phi k}) & \xrightarrow{\ j\ } & \rho^{-1}\{1,\ldots,k\} \\
\Big\downarrow \mathrm{Int}_G(g_\phi) & \nearrow_{j_k} & \\
G_{k-1} & &
\end{array}$$

where j is the inclusion of $\mathrm{Stab}_G(e_{\phi 1},\ldots,e_{\phi k})$ in $\rho^{-1}\{1,\ldots,k\}$ as group of automorphisms of $(e_{\phi 1},\ldots,e_{\phi k})$. The diagram above does not commute, but $g_\phi : (e_{\phi 1},\ldots,e_{\phi k}) \to (e_1,\ldots,e_k)$ is a morphism between the functors j and $j_k \circ \mathrm{Int}_G(g_\phi)$ as is easily checked. By I(2.1)(ii) the maps $|j|$ and $|j_k \circ \mathrm{Int}_G(g_\phi)|$ are homotopic, so

$$H_*(j_k \circ \mathrm{Int}_G(g_\phi)) = H_*(j)$$

The following diagram does commute

$$\begin{array}{ccccc}
G_{l-1} & \xrightarrow{\ j_l\ } & \rho^{-1}\{1,\ldots,l\} & \xrightarrow{\ i_l\ } & \{1,\ldots,l\}\backslash\rho \\
\Big\downarrow \mathrm{inc} & & & & \Big\downarrow \phi^* \\
\mathrm{Stab}_G(e_{\phi 1},\ldots,e_{\phi k}) & \xrightarrow{\ j\ } & \rho^{-1}\{1,\ldots,k\} & \xrightarrow{\ i_k\ } & \{1,\ldots,k\}\backslash\rho
\end{array}$$

in which $\mathrm{inc} : G_{l-1} \to \mathrm{Stab}_G(e_{\phi 1},\ldots,e_{\phi k})$ is the inclusion. It follows

$$H_*(\phi^*)H_*(i_l \circ j_l) = H_*(i_k)H_*(j)H_*(\mathrm{inc})$$

So

$$H_*(\phi^*)H_*(i_1 \circ j_1) = H_*(i_k \circ j_k)H_*(\mathrm{Int}_G(g_\phi))$$

which shows indeed that $H_*(\mathrm{Int}_G(g_\phi))$ coincides with $H_*(\phi^*)$ under our identifications.

Now we can by (2.1) compute $E^2_{pq}(F,G)$ as the homology of the complex $\mathcal{C}^{(q)}_*$ with

$$\mathcal{C}^{(q)}_p = H_q(G_p,L) \qquad p \leq d$$

$$\mathcal{C}^{(q)}_p = 0 \qquad p > d$$

and boundaries $\partial^{(q)}_p$ which are for $p = 1,\dots,d$ given by

$$\partial^{(q)}_p = \sum_{i=0}^{p} (-1)^i \, H_q(\mathrm{Int}_G(g_{\delta^p_i}))$$

Moreover, the homomorphism $H_q(G_0,L) \to H_q(G,L)$ given by

$$H_q(G_0,L) \to E^2_{pq}(F,G) \xrightarrow{\text{edge}} H_q(\mathfrak{X}(F,G),L) \xrightarrow{H_q(\pi)} H_q(G,L)$$

is easily seen to be the map coming from the inclusion $G_0 \to G$, using I §3.

3. Stability.

In the coming sections, we apply the theory developed so far to our main problem, the stability of $H_*(GL_n(R),A)$ for a commutative ring R with coefficients in a commutative ring A. As an introduction to our method, we first derive the result of Nakaoka on the stability of the homology of the symmetric groups, see [7].

(3.1) Theorem. For $n \geq 2m$, the map $H_m(S_n,A) \to H_m(S_{n+1},A)$ is an isomorphism for any commutative ring A.

Proof. We use induction on m. For $m = 0$ there is nothing to prove. Suppose we have proved the theorem for all $m' < m$, with $m \geq 1$.

We take $n \geq 2m$ and we apply the theory of sections 1 and 2 of this chapter with $G = S_{n+1}$, $F = \mathcal{O}(n+1)$. We suppress the coefficients A in our notation.

By III (2.1), $\mathcal{O}(n+1)$ is n-spherical, so because $m < n$ we have by (1.1)

$$H_m(\mathcal{K}(\mathcal{O}(n+1),S_{n+1})) \cong H_m(S_{n+1}).$$

It follows that we are through if we show that

$$H_m(S_n) \cong H_m(\mathcal{K}(\mathcal{O}(n+1),S_{n+1})).$$

Choose $(e_1,\ldots,e_{n+1}) = (1,\ldots,n+1)$. We have for $i = 0,\ldots,n$ that $G_i = \mathrm{Stab}_{S_{n+1}}(1,\ldots,i+1) \cong S_{n-i}$. Moreover, for a morphism $\phi : \{1,\ldots,k\} \to \{1,\ldots,l\}$ in $Q(n+1)$ we can take $g_\phi \in S_l$. But then g_ϕ commutes with the elements of $\mathrm{Stab}_G(1,\ldots,l) = G_{l-1}$. We conclude that $H_*(\phi^*) = H_*(\mathrm{Int}_G(g_\phi))$ is always equal to $H_*(\mathrm{inc})$ with $\mathrm{inc} : G_{l-1} \to G_{k-1}$ the inclusion map. It follows in particular for all i and k with $0 \leq i \leq k$ that $H_*((\delta_i^k)^*) = H_*(\mathrm{inc}) : H_*(G_k) \to H_*(G_{k-1})$. So the differentials of the complex $\mathcal{C}_*^{(q)}$ described at the end of section 2 are given by

$$\partial_{2i-1}^{(q)} = 0 \qquad 2i \leq n+1$$
$$\partial_{2i}^{(q)} = H_*(\mathrm{inc}) \qquad 2i \leq n$$

By section 2, we find that the E^2-term of the spectral sequence of theorem (2.3) is described as follows

$$E^2_{0q}(F,G) = H_q(S_n)$$
$$E^2_{2i,q}(F,G) = \mathrm{Ker}(H_q(S_{n-2i}) \to H_q(S_{n-2i+1})) \qquad 2 \leq 2i \leq n$$
$$E^2_{2i-1,q}(F,G) = \mathrm{Coker}(H_q(S_{n-2i}) \to H_q(S_{n-2i+1}) \qquad 2 \leq 2i \leq n.$$

In case n is odd, we find because $G_n = \{e\}$ that

$$E^2_{n0}(F,G) = \mathbb{Z}$$

$$E^2_{nq}(F,G) = 0 \qquad q > 0$$

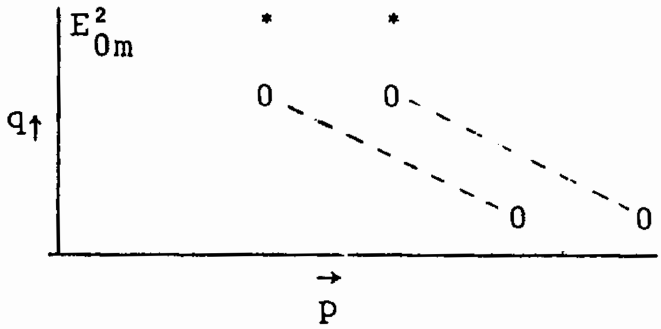
Finally, for $p > n$

$$E^2_{pq}(F,G) = 0.$$

Our induction hypothesis states that $H_q(S_{n-2i}) \to H_q(S_{n-2i+1})$ is an isomorphism as soon as $q < m$ and $2q \leqslant n-2i$, so for $i > 0$, $q < m$ and $2(i+q) \leqslant n$ we find

$$E^2_{2i,q}(F,G) = E^2_{2i-1,q}(F,G) = 0.$$

Using this, we shall now verify that $E^2_{pq}(F,G) = 0$ for $p+q=m$, $p \geqslant 1$, and for $p+q = m+1$, $p \geqslant 2$. We have $q \leqslant m-1$. If p is even, say $p = 2i$, the result follows from $2(i+q) =$ $= p+2q \leqslant 2m \leqslant n$, and if p is odd, say $p = 2i-1$ it follows from $2(i+q) = p+2q+1 \leqslant 2m \leqslant n$. So the E^2-diagram of the spectral sequence looks like



General spectral sequence theory, as exposed in Cartan-Eilenberg [1] Chap XV §5 shows that

$$\begin{aligned} H_m(S_n) &\cong E^2_{0m}(F,G) \\ &\cong E^\infty_{0m}(F,G) \\ &\cong H_m(\mathcal{K}(F,G)) \end{aligned}$$

which finishes our proof.

Let R be a commutative ring. We want to use the method of (3.1) to consider the relation between $H_m(GL_n(R),A)$ and $H_m(GL_{n+1}(R),A)$ for a commutative ring A. So we take

$G = GL_{n+1}(R)$, $F = \mathcal{O}(n+1,R)$. We know that G acts transitively on F. Take $(e_1,\ldots,e_{n+1}) \in F$. Further notations are as explained in sections 1 and 2. We have

$$G_i = \{g \in GL_{n+1}(R) \mid g(e_1,\ldots,e_{i+1}) = (e_1,\ldots,e_{i+1})\}$$

i.e. G_i consists of partitioned matrices of the form

$$i+1\left\{\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & * \\ 0 & & 1 & \\ \hline & 0 & & * \end{array}\right)\right.$$

So G_i is the semi-direct product $GL_{n-i}(R) \ltimes (R^{n-i})^{i+1}$ of $GL_{n-i}(R)$ and $i+1$ copies of R^{n-i}. We denote this group by $GA^{i+1}_{n-i}(R)$. Observe that $GA_n(R) = GA^1_n(R)$. We embed $GL_{n-i}(R)$ in $GA^{i+1}_{n-i}(R)$ by sending

$$g \mapsto i+1\left\{\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \\ \hline & 0 & & g \end{array}\right)\right.$$

Let $\phi : \{1,\ldots,k\} \to \{1,\ldots,l\}$ be a morphism in $Q(n+1)$. We can take $g_\phi \in GL_{n+1}(R)$ of the form

$$l\left\{\left(\begin{array}{c|ccc} h_\phi & & 0 & \\ \hline & 1 & & 0 \\ 0 & & \ddots & \\ & 0 & & 1 \end{array}\right)\right.$$

with $h_\phi \in GL_l(R)$, so that g_ϕ permutes $e_1,\ldots,e_l$ and fixes $e_{l+1},\ldots,e_{n+1}$. In the proof of Nakaoka's theorem, we used that $H_*(\phi^*) = H_*(\mathrm{int}_G(g_\phi)) = H_*(\mathrm{inc})$, but unfortunately $\mathrm{Int}_G(g_\phi)$ does not fix all elements of G_{l-1} in the present situation. However, g_ϕ does commute with the image of $GL_{n-l+1}(R)$ in $G_{l-1} = GA^{l}_{n-l+1}(R)$. So as soon as $H_q(GA^l_{n-l+1}(R),A)$ is isomorphic to $H_q(GL_{n-l+1}(R),A)$ we find $H_q(\phi^*) = H_q(\mathrm{inc})$, so $H_q(\phi^*)$ is independent of ϕ. Hence we shall need that

$H_q(GA_n^k(R),A) \cong H_q(GL_n(R),A)$ for several values of q and n. We take this condition for granted in proposition (3.3). We shall consider it in detail in section 4. For the sake of clarity, we introduce

(3.2) Definition. Let R be a commutative ring, $e \geq 0$ an integer. For homology with coefficients in a commutative ring A define the statements (α_m) and (β_m) as follows:

(α_m) $H_m(GL_n(R),A) \to H_m(GL_{n+1}(R),A)$ is an isomorphism for $n \geq 2m+e$ and a surjection for $n \geq 0$, $n \geq 2m+e-1$.

(β_m) $H_m(GL_n(R),A) \to H_m(GA_n^k(R),A)$ is an isomorphism for $n \geq 0$, $n \geq 2m+e-1$, and all k.

(3.3) Proposition. Let R be a commutative ring such that $\mathcal{O}(n,R)$ is n-1-spherical for all n. Let A be a commutative ring, $e,f \geq 0$ integers. Assume (α_m) holds for $m < f$, and (β_m) holds for $m \leq f$. In the case $e = 0$, $f = 1$ assume moreover that $H_1(GL_1(R),A) \to H_1(GL_2(R),A)$ is surjective. Then (α_f) holds.

Proof. We suppress both R and A in our notation, and proceed as in the proof of Nakaoka's theorem. For $f = 0$ we have nothing to prove, so suppose $f \geq 1$.

We want to apply the theorem of sections 1 and 2 of this chapter with $G = GL_{n+1}$, $F = \mathcal{O}(n+1,R)$.

By assumption, $\mathcal{O}(n+1,R)$ is n-spherical, so for $n \geq 2f+e$

$$H_f(\ast(F,G)) \cong H_f(GL_{n+1}).$$

For $n = 2f+e-1$ we find, because $f \leq n$, that

$$H_f(\ast(F,G)) \twoheadrightarrow H_f(GL_{n+1}).$$

So we have to show for $n \geq 2f+e$ that

$$H_f(GL_n) \cong H_f(\ast(F,G))$$

and for $n = 2f+e-1$ that

$$H_f(GL_n) \twoheadrightarrow H_f(*(F,G))$$

Let $(e_1,\ldots,e_{n+1})$ be the standard basis in R^{n+1}. We already found

$$G_p = GA^{p+1}_{n-p}$$

By assumption, we have for $q \leqslant f$, $n-p \geqslant 2q+e-1$ that

$$\mathcal{C}^{(q)}_p = H_q(G_p) \cong H_q(GL_{n-p})$$

and so for i,p and q with $0 \leqslant i \leqslant p$, $q \leqslant f$, $2q \leqslant n-p-e+1$ we have $H_q((\delta^p_i)^*) = H_q(\mathrm{inc}) : H_q(G_p) \to H_q(G_{p-1})$ by the remarks above. So we find for the differentials $\partial^{(q)}_p$ that $\partial^{(q)}_p = 0$ if p is odd, $q \leqslant f$, $2q \leqslant n-p-e+1$ and $\partial^{(q)}_p = H_q(\mathrm{inc})$ if p is even, $p \leqslant n$, $q \leqslant f$, $2q \leqslant n-p-e+1$.

It follows that we have for $q \leqslant f$, $1 \leqslant 2i \leqslant n-2q-e$

$$E^2_{2i,q}(F,G) = \mathrm{Ker}(H_q(GL_{n-2i}) \to H_q(GL_{n-2i+1}))$$

and for $q \leqslant f$, $1 \leqslant 2i-1 \leqslant n-2q-e$, $2i-1 \neq n$

$$E^2_{2i-1,q}(F,G) = \mathrm{Coker}(H_q(GL_{n-2i}) \to H_q(GL_{n-2i+1})).$$

So we find for $q < f$, $1 \leqslant p \leqslant n-2q-e$ and $p \neq n$ if p is odd that

$$E^2_{pq}(F,G) = 0$$

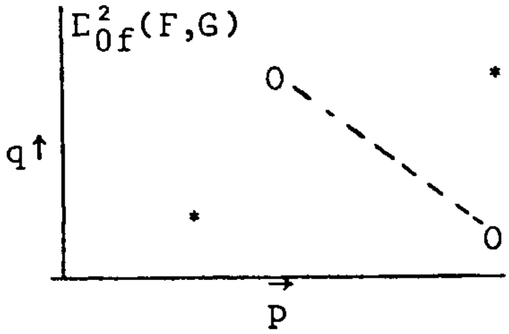
since (α_q) holds for $q < f$. Moreover, for $q \leqslant f$, $2q \leqslant n-e+1$ we have

$$H_q(GL_n) \twoheadrightarrow E^2_{0q}(F,G)$$

and for $q \leqslant f$, $2q \leqslant n-e$, even

$$H_q(GL_n) \cong E^2_{0q}(F,G).$$

Assume if $e = 0$, $f = 1$ that $n \neq 1$, and let $n \geqslant 2f+e-1$. Take $p \geqslant 1$, $p+q = f$, then $p \leqslant n-1$ and $p = f-q \leqslant n-e-2q$ so $E^2_{pq}(F,G) = 0$. Hence the E^2-diagram of the spectral sequence looks like



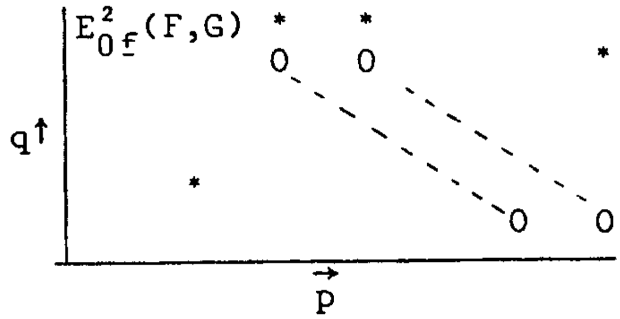
so

$$E^2_{0f}(F,G) \twoheadrightarrow H_f(\mathfrak{X}(F,G)) \twoheadrightarrow H_f(GL_{n+1})$$

We conclude for $n \geqslant 2f+e-1$ that

$$H_f(GL_n) \twoheadrightarrow H_f(GL_{n+1}).$$

Assume now $n \geqslant 2f+e$. Let $p \geqslant 2$, $p+q = f+1$. Then $p \leqslant n-1$ if p is odd, and $p = f+1-q \leqslant n-e-2q$, so again $E^2_{pq}(F,G) = 0$. So in this case the E^2-diagram looks like



We conclude that

$$H_f(GL_n) \cong E^2_{0f}(F,G)$$
$$\cong E^\infty_{0f}(F,G)$$
$$= H_f(\mathfrak{X}(F,G)).$$

So for $n \geqslant 2f+e$

$$H_f(GL_n) \xrightarrow{\sim} H_f(GL_{n+1})$$

which finishes our proof.

4. Stability continued.

To get stability for the homology of $GL_n(R)$ we must prove that $H_q(GA^k_n(R),A) \cong H_q(GL_n(R),A)$ for n and q within a certain range. We shall show that this holds in case both $\Theta(n,R)$ and $\mathcal{H}(n,k,R)$ are CM-posets. We begin this section by introducing certain groups $GA_n^{k,l}(R)$, which occur as stabilizers when we try to prove (β_m) (cf.(3.2),(3.3)).

Let R be a ring. Then $GA_n^k(R)$ is a subgroup of $GL_{n+k}(R)$ so we can form $GA_n^k(R) \ltimes (R^{n+k})^l$. It is a subgroup of $GL_{n+k}(R) \ltimes (R^{n+k})^l = GA_{n+k}^l(R)$, and we denote it by $GA_n^{k,l}(R)$. We always think of $GA_n^{k,l}(R)$ as the subgroup of $GL_{n+k+l}(R)$, consisting of partitioned matrices of the form

$$\begin{array}{c} k \\ \\ n \\ \\ l \end{array} \left(\begin{array}{c|c|c} \begin{matrix} 1 & & \theta \\ & \ddots & \\ \theta & & 1 \end{matrix} & * & * \\ \hline \theta & * & * \\ \hline \theta & \theta & \begin{matrix} 1 & & \theta \\ & \ddots & \\ \theta & & 1 \end{matrix} \end{array}\right)$$

We obviously have embeddings

$$GA_n^{k,l}(R) \to GA_n^{k+1,l}(R)$$

given by

$$g \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g \end{array}\right)$$

and

$$GA_n^{k,l}(R) \to GA_n^{k,l+1}(R)$$

given by

$$g \mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right)$$

It follows

$$GA_n^{k+1,l}(R) \cong GA_n^{k,l}(R) \ltimes R^{n+l}$$
$$GA_n^{k,l+1}(R) \cong GA_n^{k,l}(R) \ltimes R^{n+k}$$

Furthermore, the groups $GA_n^{k,l}(R)$ enjoy the following property

(4.1) <u>Lemma</u>. There are isomorphisms

$$\tau_n^{k,l} : GA_n^{k,l}(R) \to GA_n^{l,k}(R)$$

such that the diagram below commutes

$$\begin{array}{ccc} GA_n^{k,l}(R) & \longrightarrow & GA_n^{k,l+1}(R) \\ \Big\downarrow \tau_n^{k,l} & & \Big\downarrow \tau_n^{k,l+1} \\ GA_n^{l,k}(R) & \longrightarrow & GA_n^{l+1,k}(R) \end{array}$$

Proof. Let $c_n \in GL_n(R)$ be the $n \times n$ permutation matrix given by $c_n(e_i) = e_{n-i+1}$. Define the automorphism τ_n of $GL_n(R)$ by $\tau_n(g) = \mathrm{Int}(c_n)({}^t g^{-1})$ for $g \in GL_n(R)$, where t denotes taking transpose. Let $\tau_n^{k,l}$ be the restriction of τ_{n+k+l} to $GA_n^{k,l}(R)$. We leave the remainder of the proof as an exercise.

To describe the homology of $GA_n^{k,l+1}(R)$, we make it act on the poset $\mathcal{A}(n+k,k,R)$. To do this, we project first $GA_n^{k,l+1}(R)$ onto $GA_n^{k,l}(R)$ by suppressing l copies of R^{n+k}. Now $GA_n^{k,1}(R)$ is a subgroup of $GA_{n+k}^{1}(R) = GA_{n+k}(R)$. The latter acts on $\mathcal{A}(n+k,R)$, and $\mathcal{A}(n+k,k,R)$ is a subposet of $\mathcal{A}(n+k,R)$. So we have to prove for $g \in GA_n^{k,1}(R)$ that $g\mathcal{A}(n+k,k,R) \subset \mathcal{A}(n+k,k,R)$. Write $g = (g_0,a)$ with $g_0 \in GA_n^k(R)$, $a \in R^{n+k}$. Take $(v_0,\ldots,v_p) \in \mathcal{A}(n+k,k,R)$, then $(v_1-v_0,\ldots,v_p-v_0) \in \mathcal{O}(n+k,k,R)$. Moreover

$$g(v_0,\ldots,v_p) = (g_0v_0+a,\ldots,g_0v_p+a)$$

which is in $\mathcal{A}(n+k,k,R)$ if we show that $(g_0(v_p-v_0),\ldots,g_0(v_p-v_0)) \in \mathcal{O}(n+k,k,R)$. But this is obviously true as $g_0(e_j) = e_j$ for $j = 1,\ldots,k$.

(4.2) Lemma. $GA_n^{k,l+1}(R)$ acts transitively on $\mathcal{A}(n+k,k,R)$ for all $l \geq 0$.

Proof. We may take $l = 0$. Let $e_0 = 0$, then $(e_0,e_{k+1},\ldots,e_{k+n}) \in \mathcal{A}(n+k,k,R)$. Let $(v_0,\ldots,v_n) \in \mathcal{A}(n+k,k,R)$ be arbitrary. Define $g \in GL_{n+k}(R)$ by $ge_j = e_j$, $j = 1,\ldots,k$,

$ge_j = v_{j-k} - v_0$, $j = k+1,\dots,k+n$. Then obviously $(g,v_0) \in GA_n^{k,1}(R)$, and

$$(g,v_0)(e_0,e_{k+1},\dots,e_{k+n}) = (v_0,\dots,v_n)$$

We introduce now a statement (γ_m) which will trivially imply (β_m). Let R,A,e be as in (3.2) and define:

(γ_m) $H_m(GL_n(R),A) \to H_m(GA_n^{k,1}(R),A)$ is an isomorphism for $n \geq 0$, $n \geq 2m+e-1$ and all k,1.

The next result is the last preparation for our main stability theorem.

(4.3) Proposition. Let R be a commutative ring, such that $\mathcal{A}(n+k,k,R)$ is n-spherical for all n and k. Let A be a commutative ring, e,f $\geq$ 0 integers. Assume (α_m) and (γ_m) hold for $m < f$. In case $e = 0$, $f = 1$, assume moreover that $H_1(GL_1(R),A) \to H_1(GA_1^{k,1}(R),A)$ is an isomorphism for all k and 1. Then (γ_f) holds.

Proof. We suppress R and A again in our notation. Since for $f = 0$ we have nothing to prove, assume $f \geq 1$. We proceed as in the proof of (3.1) and (3.3), considering the action of $G = GA_n^{k,1+1}$ with $1 \geq 0$ on $F = \mathcal{A}(n+k,k,R)$. Further notations are as in sections 1 and 2.

Since $2f \leq n+1$ we certainly have $f \leq n$, so because $\mathcal{A}(n+k,k,R)$ is n-spherical we find by (1.1)

$$H_f(\mathfrak{X}(F,G)) \twoheadrightarrow H_f(GA_n^{k,1+1})$$

Fix the element $(e_0,e_{k+1},\dots,e_{k+n}) \in \mathcal{A}(n+k,k,R)$ with $e_0 = 0$. Then

$$G_0 = \mathrm{Stab}_G(e_0) = GA_n^{k,1}$$
$$= GA_n^k \ltimes (R^{n+k})^1.$$

The elements of $(R^{n+k})^1$ act trivially on $\mathcal{A}(n+k,k,R)$ so for $i \geqslant 1$ we find

$$\begin{aligned} G_i &= \mathrm{Stab}_{GA_n^k}(e_{k+1},\dots,e_{k+i}) \ltimes (R^{n+k})^1 \\ &= GA_{n-i}^{k+i} \ltimes (R^{n+k})^1 \\ &= GA_{n-i}^{k+i,1} \end{aligned}$$

The group GL_{n-i} is embedded in $G_i = GA_{n-i}^{k+i,1}$ as group of automorphisms of $< e_{k+i+1},\dots,e_{k+n} >$.

Let ϕ be a morphism $\{1,\dots,s\} \to \{1,\dots,t\}$ in $Q(n+1)$. We can take $g_\phi \in GA_n^{k,1}$ such that g_ϕ fixes e_i for $i = 1,\dots,k,k+t,\dots,k+n$ and permutes $e_0,e_{k+1},\dots,e_{k+t-1}$. Hence g_ϕ commutes with the elements of GL_{n-t+1}. It follows that $H_m(\phi^*)$ is independent of ϕ as soon as $H_m(GA_{n-t+1}^{k+t-1,1}) \cong H_m(GL_{n-t+1})$, which happens by assumption if $m < f$, $2m \leqslant n-t-e+2$.

We conclude for $q < f$, $p \leqslant n$, $2q \leqslant n-p-e+1$ that

$$\mathcal{C}_p^{(q)} \cong H_q(GL_{n-p}).$$

Moreover, we find for the differentials $\partial_p^{(q)}$ that $\partial_p^{(q)} = 0$ if p is odd, $q < f$, $2q \leqslant n-p-e+1$, and $\partial_p^{(q)} = H_q(\mathrm{inc})$ if p is even, $p \leqslant n$, $q < f$, $2q \leqslant n-p-e+1$.

By assumption we know for $m < f$, $2m \leqslant n-e$ that

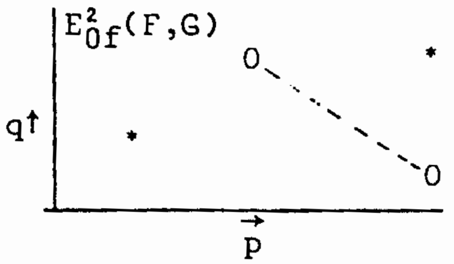
$$H_m(GL_n) \xrightarrow{\sim} H_m(GL_{n+1})$$

and for $m < f$, $2m = n-e+1$ that

$$H_m(GL_n) \twoheadrightarrow H_m(GL_{n+1})$$

As in the proof of (3.3) we conclude for $q < f$, $p \leqslant n-1$ if p is odd and $1 \leqslant p \leqslant n-2q-e$ that $E^2_{pq}(F,G) = 0$.

Assume now $n \geqslant 2f+e-1$ and if $e = 0$, $f = 1$ that $n \geqslant 2$. Then if $p \geqslant 1$, $p+q = f$ we have $p \leqslant n-1$ and $p \leqslant n-2q-e$, so $E^2_{pq}(F,G) = 0$. Hence the E^2-diagram again looks like



We conclude

$$H_f(GA_n^{k,l}) \twoheadrightarrow E^2_{0f}(F,G) \twoheadrightarrow E^\infty_{0f}(F,G)$$

and

$$E^\infty_{0f}(F,G) \twoheadrightarrow H_f(\mathcal{X}(F,G)).$$

We find

$$H_f(GA_n^{k,l}) \twoheadrightarrow H_f(GA_n^{k,l+1}).$$

As the inclusion $GA_n^{k,l} \to GA_n^{k,l+1}$ splits, it follows

$$H_f(GA_n^{k,l}) \cong H_f(GA_n^{k,l+1}).$$

Using (4.1) this gives

$$H_f(GA_n^{k,l}) \cong H_f(GA_n^{k+1,l}).$$

Inductively, one finds for $n \geqslant 2f+e-1$ that

$$H_f(GA_n^{k,l}) \cong H_f(GL_n)$$

that is, (γ_f).

We shall now derive our main stability theorem from (3.3) and (4.3).

(4.4) Theorem. Let R be a commutative ring, such that $\mathcal{O}(n,R)$ is n-1-spherical for all n, and $\mathcal{H}(n+k,k,R)$ is n-spherical for all n and k. Let A be a commutative ring. Take e = 0 if

$$H_1(GL_1(R),A) \cong H_1(GA_1^{k,l}(R),A)$$

$$H_1(GL_1(R),A) \twoheadrightarrow H_1(GL_2(R),A)$$

and e = 1 if not. Then

(i) for $n \geqslant 0$, $n \geqslant 2m+e-1$, all k,l

$$H_m(GL_n(R),A) \cong H_m(GA_n^{k,l}(R),A)$$

(ii) for $n \geqslant 2m+e$

$$H_m(GL_n(R),A) \xrightarrow{\sim} H_m(GL_{n+1}(R),A)$$

(iii) for $n \geqslant 0$, $n = 2m+e-1$

$$H_m(GL_n(R),A) \twoheadrightarrow H_m(GL_{n+1}(R),A).$$

Proof. Induction on m. For $m = 0$, we have nothing to prove, so suppose we have proved the result for $0 \leqslant m < f$. This is equivalent to saying that (α_m) and (γ_m) hold for $m < f$. Then (4.3) implies that (γ_f), that is, (i) holds for $m = f$. Since (γ_m) implies (β_m), we can apply (3.3) to conclude that (α_f) holds, i.e. (ii) and (iii) hold for $m = f$.

5. Stability concluded.

In this section, we want to combine the results of chapters III and IV. First we give conditions implying $H_1(GL_1(R),A) \twoheadrightarrow H_1(GL_2(R),A)$ and $H_1(GL_1(R),A) \cong H_1(GA_1^{k,l}(R),A)$.

(5.1) Lemma. Let R be a commutative ring, J an ideal contained in its Jacobson radical. Assume R/J is Euclidean. Let A be a commutative ring. Then suppose either $1-R^*$ generates the unit ideal or $\frac{1}{2} \in A$. Then for all k,l

$$H_1(GL_1(R),A) \cong H_1(GA_1^{k,l}(R),A)$$

$$H_1(GL_1(R),A) \twoheadrightarrow H_1(GL_2(R),A).$$

Proof. Denote for $t \in R$, $i \neq j$ by $E_{ij}(t)$ the matrix

$$i \begin{pmatrix} 1 & & \overset{j}{\vdots} & \\ \cdots & \ddots & t & 0 \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

Then $H_1(GA_1^{k,1}(R),\mathbb{Z})$ is generated by the images of the $E_{ij}(t)$ with $i = k+1$, $j \geq k+2$ or $j = k+1$, $i \leq k$, and $H_1(GL_1(R),\mathbb{Z})$. By conjugating these $E_{ij}(t)$ by matrices

$$k+1 \begin{pmatrix} 1 & & & & & \\ & \ddots & & & \Theta & \\ & & 1 & & & \\ - & - & - & u & & \\ & & & & 1 & \\ & \Theta & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

with $u \in R^*$ we find that the image of $E_{ij}(t)$ in $H_1(GA_1^{k,1}(R),A)$ equals the image of $E_{ij}(vt)$ for all $v \in R^*$. So if $1-R^*$ generates the unit ideal, $E_{ij}(t)$ goes to zero in $H_1(GA_1^{k,1}(R),A)$, and in general, taking $v = -1$, we find that the image of $E_{ij}(t)$ is 2-torsion. We conclude that $H_1(GL_1(R),\mathbb{Z}) \cong H_1(GA_1^{k,1}(R),\mathbb{Z})$ if $1-R^*$ generates the unit ideal, and that $H_1(GA_1^{k,1}(R),\mathbb{Z})$ is generated by $H_1(GL_1(R),\mathbb{Z})$ and 2-torsion elements in the general case.

As R/J is Euclidean, we have $SL_2(R) = E_2(R)$ and so

$$H_1(GA_1^1(R),\mathbb{Z}) \twoheadrightarrow H_1(GL_2(R),\mathbb{Z}).$$

So if $1-R^*$ generates the unit ideal, we find

$$H_1(GL_1(R),\mathbb{Z}) \twoheadrightarrow H_1(GL_2(R),\mathbb{Z})$$

and in the general case $H_1(GL_2(R),\mathbb{Z})$ is generated by the image of $H_1(GL_1(R),\mathbb{Z}) \to H_1(GL_2(R),\mathbb{Z})$ and 2-torsion elements.

For all groups G we have

$$H_1(G,A) \cong H_1(G,\mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

Since tensoring with an A such that $\frac{1}{2} \in A$ kills 2-torsion, we are through.

From III (6.2) and (4.4), (5.1) of this chapter we find

(5.2) Theorem. Let R be a commutative ring; J is an ideal contained in its Jacobson radical. Let A be a commutative ring. Assume R is local or R/J is a subring of $\mathbb{Q}$. Let e = 0 if $1-R^*$ generates the unit ideal or $\frac{1}{2} \in A$, and e = 1 if not. Then for $n \geq 0$, $n \geq 2m+e-1$ and all k,l

$$H_m(GL_n(R),A) \cong H_m(GA_n^{k,l}(R),A)$$

for $n \geq 2m+e$

$$H_m(GL_n(R),A) \xrightarrow{\sim} H_m(GL_{n+1}(R),A)$$

and for $n \geq 0$, $n = 2m+e-1$

$$H_m(GL_n(R),A) \twoheadrightarrow H_m(GL_{n+1}(R),A).$$

The next result tells when $1-R^*$ generates the unit ideal.

(5.3) Proposition (i). Let R be local. Then $1-R^*$ generates the unit ideal if and only if the residue class field is not $\mathbb{F}_2$.
(ii) Let R be a ring, J an ideal contained in its Jacobson radical. Assuem R/J is a subring of $\mathbb{Q}$. Then $1-R^*$ generates the unit ideal if and only if $\frac{1}{2} \in R/J$.
Proof. This is left as an exercise.

For a direct application of (3.3) we have to prove the condition (β_m) on the homology of $GA_n^k(R)$. Scrutinizing the proof of theorem 1' of Quillen [10] yields

(5.4) Theorem. Let R be a commutative ring, F a field. Assume if char F = 0 there exists a prime number invertible in R and if char F = p > 0 that p is invertible in R. Then for all n and k

$$H_*(GL_n(R),F) \cong H_*(GA_n^k(R),F).$$

Combining III (6.2) with (3.3) and (5.4) yields

(5.5) <u>Theorem</u>. Let R be a commutative ring, J an ideal contained in its Jacobson radical such that R/J is Euclidean. Let F be a field. Assume if char F = 0 there exists a prime number invertible in R and if char F = p > 0 that p is invertible in R. Then for $n \geq 2m$

$$H_m(GL_n(R),F) \xrightarrow{\sim} H_m(GL_{n+1}(R),F)$$

and for $n \geq 0$, $n = 2m-1$

$$H_m(GL_n(R),F) \twoheadrightarrow H_m(GL_{n+1}(R),F).$$

<u>Proof</u>. Since either $\frac{1}{2} \in F$ or $\frac{1}{2} \in R$, which implies that $1-R^*$ generates the unit ideal, we can take e = 0 in (3.3) by (5.1).

<u>Examples</u>. 1) Let r be an integer, R a subring of $\mathbb{Q}$. Let e = 0 if $\frac{1}{2} \in R$, e = 1 if not. Then for $n \geq 2m+e$

$$H_m(GL_n(R[X]/(X^r)),\mathbb{Z}) \xrightarrow{\sim} H_m(GL_{n+1}(R[X]/(X^r)),\mathbb{Z})$$

and for $n = 2m+e-1$

$$H_m(GL_n(R[X]/(X^r)),\mathbb{Z}) \twoheadrightarrow H_m(GL_{n+1}(R[X]/(X^r)),\mathbb{Z}).$$

Observe that we have, by taking $R = \mathbb{Z}$, r = 0, m = 2

$$H_2(GL_4(\mathbb{Z}),\mathbb{Z}) \twoheadrightarrow H_2(GL_5(\mathbb{Z}),\mathbb{Z}) \xrightarrow{\sim} H_2(GL_6(\mathbb{Z}),\mathbb{Z}) \to \dots$$

which is a known result, cf. Milnor [6].

2) Let R be a $\mathbb{Q}$-algebra. By a universal coefficient theorem argument, it follows from (5.4) that

$$H_*(GA_n^k(R),\mathbb{Z}) \cong H_*(GL_n(R),\mathbb{Z}).$$

If J is an ideal contained in the Jacobson radical of R such that R/J is Euclidean, we conclude from III (6.2), (3.3) and (5.1) because $\frac{1}{2} \in R$ that for $n \geq 2m$

$$H_m(GL_n(R),\mathbb{Z}) \xrightarrow{\sim} H_m(GL_{n+1}(R),\mathbb{Z})$$

and for $n = 2m-1$

- $H_m(GL_n(R),\mathbb{Z}) \twoheadrightarrow H_m(GL_{n+1}(R),\mathbb{Z})$

We can take for instance $R = \mathbb{Q}[X][[Y]]$ or $R = \mathbb{Q}[X,Y]/(Y^r)$ for some natural number r.

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Samenvatting.

Het hele gebeuren rond een promotie brengt met zich mee dat de voor die gelegenheid geproduceerde wiskundige tekst ook buiten de eng-wiskundige kring verspreiding vindt. Wellicht dat U, geachte niet-wiskundige lezer, de vraag voelt opkomen wat het thans voor U liggende geschrift eigenlijk behelst. Gaarne wil ik U hier in deze samenvatting iets over vertellen.

Welnu, ik ben uitgegaan van een concreet probleem, afkomstig uit wat in niet-vakkringen hogere wiskunde genoemd wordt. Met mathematisch kunst- en vliegwerk heb ik dit probleem omgezet in een technische vraag die met -wiskundig gezien- eenvoudige middelen kon worden opgelost.

Deze laatste vraag betreft de vorm der dingen. Laat ik het soort probleem toelichten aan de hand van een aardrijkskundig voorbeeld. Stel eens, dat U op een eiland zit. De onderstaande eilanden zijn in zekere zin hetzelfde

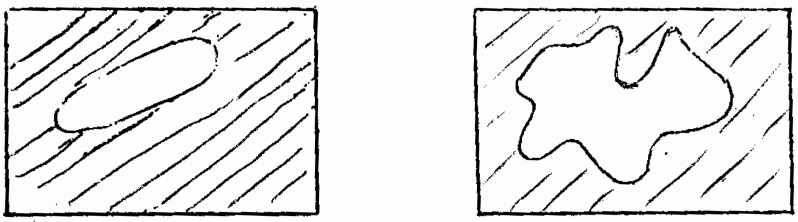


fig. I. Gearceerd is water.

en de volgende ook

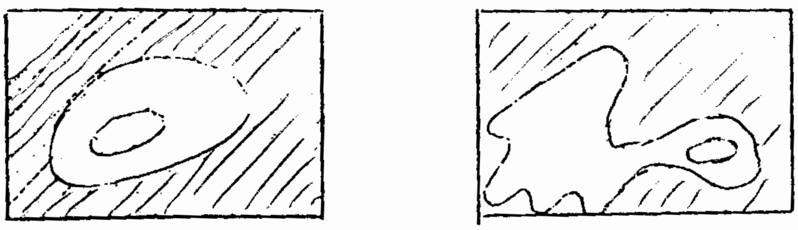
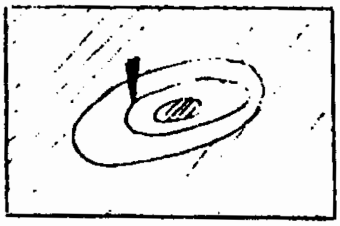


fig. II.

De eilanden van fig.II verschillen echter met die van fig.I door de aanwezigheid van een binnenmeer.

We vragen ons nu af hoe U als eilandbewoner de aanwezigheid van zo'n meer kunt bepalen, als U alleen maar over land mag lopen. U zou dat als volgt kunnen doen, zie fig. III.



Sla een paal en bind daar een stuk touw aan vast. Maak, het touw afwikkelend, een rondwandeling over het eiland. Als U weer bij de paal bent aangekomen, trekt U het touw aan. Kijk of dat kan zonder dat het touw nat wordt. Zo ja, dan bent U niet om een meer heengewandeld, zo nee, dan bent U wel om een meer heen gewandeld.

Deze vraag, geachte lezer, heb ik aangepakt in een algemener en wiskundiger jasje. Laat ik tenslotte voor de enigszins wiskundig onderlegden het probleem formuleren.

Beschouw $GL_n(\mathbb{Z})$, de groep van de inverteerbare $n \times n$-matrices met gehele coëfficienten. Stop $GL_n(\mathbb{Z})$ in $GL_{n+1}(\mathbb{Z})$ door rechtsonder een 1 te schrijven en verder 0. We vragen ons af: stabiliseert de homologie van $GL_n(\mathbb{Z})$ met gehele coefficienten, d.w.z. is voor vaste m de afbeelding

$$H_m(GL_n(\mathbb{Z}),\mathbb{Z}) \to H_m(GL_{n+1}(\mathbb{Z}),\mathbb{Z})$$

een isomorfisme voor n groot? Het antwoord op deze vraag blijkt ja te zijn, als $n > 2m$.

We brengen in het proefschrift het probleem terug tot de volgende vraag: is een expliciet gegeven topologische ruimte X n-spherisch, d.w.z. is $\tilde{H}_i(X,\mathbb{Z}) = 0$ voor $i \neq n$, en is $\widetilde{H}_n(X,\mathbb{Z})$ een vrij $\mathbb{Z}$-moduul? Deze vraag kunnen we met wat ik zou willen omschrijven als een "schaar-en-lijmpot methode" bevestigend beantwoorden.

In ons aardrijkskundig voorbeeld hebben we de eigenschap: "alle wegen in X zijn samentrekbaar in X" beschouwd. Wiskundig gezegd: $\pi_1(Z) = 0$. Hieruit volgt dan weer dat $\widetilde{H}_1(X,\mathbb{Z}) = 0$. Tenslotte zij opgemerkt dat er $\pi_i(X)$ met $i > 1$ bestaan, die op een analoge wijze als $\pi_1(X)$ beschreven kunnen worden, en die nauw samenhangen met $\widetilde{H}_i(X,\mathbb{Z})$.

Voor een preciesere beschrijving van de hier geschetste methode wordt men verwezen naar de Engelse inleiding.

Curriculum vitae.

De auteur van dit proefschrift werd op 22 kuni 1951 te Rotterdam geboren. Hij bezocht de lagere school in Rotterdam, Leusden en Amersfoort. In 1964 ging hij naar de Rijks H.B.S. te Amersfoort. Aan deze school deed hij in 1969 eindexamen H.B.S.-B.

Datzelfde jaar ging hij in Utrecht natuurkunde studeren. In 1971 veranderde hij zijn hoofdvak in wiskunde. Hij volgde de colleges van onder andere de hoogleraren Van der Blij, Freudenthal, Van der Sluis, Springer en Veldkamp. In 1973 sloot hij zijn studie af met het doctoraal examen.

Vanaf 1971 is hij werkzaam bij het Mathematisch Instituut te Utrecht, tot 1973 als assistent, sinds 1973 als wetenschappelijk medewerker.

Door het bijwonen van het college en seminarium K-theorie dat in 1973/74 door zijn promotor Strooker werd gegeven, is zijn belangstelling voor de algebraische K-theorie gewekt. Uit gezamenlijk werk met Jan Stienstra resulteerde een artikel getiteld "K_2 of split radical pairs". Het onderzoek dat tot dit proefschrift heeft geleid is voortgekomen uit een bijna argeloos gestelde vraag van Strooker.

Hij heeft belangstelling voor de antieke en middeleeuwse geschiedenis. Verder is hij actief als genealoog en fuchsiakweker.