Solution of Exercise 1.15. Given $x \in \mathbf{R}^n$, the equality $y = f(x) \in \mathbf{R}^n$ yields

$$\|y\|^2 = \|f(x)\|^2 = \frac{\|x\|^2}{r^2 - \|x\|^2}, \qquad \text{i.e.} \qquad r^2\|y\|^2 - \|y\|^2\|x\|^2 = \|x\|^2;$$

therefore solving for $\|x\|$ gives

$$(\star) \qquad \|x\| = \frac{r\|y\|}{\sqrt{1 + \|y\|^2}}.$$

We now prove the injectivity of f. Indeed, suppose $f(x) = f(\widetilde{x}) = y$, for x and $\widetilde{x} \in \mathbf{R}^n$. Then $(\star)$ implies $\|x\| = \|\widetilde{x}\|$ and thus

$$\frac{1}{\sqrt{r^2 - \|x\|^2}}\, x = \frac{1}{\sqrt{r^2 - \|x\|^2}}\, \widetilde{x}, \qquad \text{so} \qquad x = \widetilde{x}.$$

Next comes the surjectivity. Given $y \in \mathbf{R}^n$ consider the equation $y = f(x)$ for $x \in \mathbf{R}^n$. Then, on the basis of $(\star)$,

$$x = \sqrt{r^2 - \|x\|^2}\, y = \sqrt{r^2 - \frac{r^2\|y\|^2}{1 + \|y\|^2}}\, y = \frac{r}{\sqrt{1 + \|y\|^2}}\, y,$$

where x belongs to B as required. Accordingly, $f : B \to \mathbf{R}^n$ is a bijection with f^{-1} as given. Finally, f and f^{-1} are continuous, being the composition of such mappings.