Solution of Exercise 2.25. Consider $g : U \setminus \{a\} \to \mathbf{R}^p$ given by $g(x) = f(x) - Lx$, then $Dg(x) = Df(x) - L \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}^p)$. In view of $\lim_{x \to a} Df(x) - L = 0$, there exists, given arbitrary $\epsilon > 0$, an open ball $V \subset \mathbf{R}^n$ satisfying $a \in V \subset U$, such that $\xi \in V \setminus \{a\}$ implies

$$\|Df(\xi) - L\|_{\mathrm{Eucl}} < \epsilon.$$

Because $V \setminus \{a\}$ may be covered by finitely many convex open sets, the Mean Value Theorem 2.5.3 applies with the mapping g and the open set $V \setminus \{a\}$. Accordingly one has, for any two points x and $x' \in V \setminus \{a\}$,

$$\|g(x) - g(x')\| \le \sup_{\xi \in V \setminus \{a\}} \|Df(\xi) - L\|_{\mathrm{Eucl}} \|x - x'\| < \epsilon \|x - x'\|.$$

This shows that g and consequently $f = g + L$ are uniformly continuous on $V \setminus \{a\}$. The mapping f can be extended continuously to all of V according to Exercise 1.33.(ii); in particular, $f(a)$ may be defined in such a way that f is continuous on U. Application of the preceding argument with x' replaced by a now gives

$$\|f(x) - f(a) - L(x - a)\| < \epsilon \|x - a\|.$$

On account of Definition 2.2.2 one sees that f is differentiable at a, and Lemma 2.2.3 then implies that $L = Df(a)$.