Solution of Exercise 2.39.

(i) Obvious from $A_i x = \sum_{1 \le j \le n} a_{ij} x_j$.

(ii) The first assertion follows by repeated application of $DA(x) = A$, for all $x \in \mathbf{R}^n$, see Example 2.2.5. The differentiability of A^{-1} can be deduced from Formula (2.7) and its subsequent sentence.

(iii) Direct consequence of the chain rule.

(iv) Note that $g \circ A(x) = g(A_1 x, \dots, A_n x)$. Therefore the chain rule and part (i) imply

$$D_k(g \circ A)(x) = \sum_{1 \le i \le n} D_i g(Ax)\, a_{ik} = \sum_{1 \le j \le n} a_{jk} (D_j g) \circ A(x).$$

Application of this identity with g replaced by $D_j g$, for $1 \le j \le n$, leads to

$$\begin{aligned} \Delta g \circ A(x) &= \sum_{1 \le k \le n} \Big(\sum_{1 \le j \le n} a_{jk} \sum_{1 \le i \le n} a_{ik}\, (D_i D_j g) \circ A(x) \Big) \\ &= \sum_{1 \le i,j \le n} \Big(\sum_{1 \le k \le n} a_{ik} a_{jk} \Big) (D_i D_j g) \circ A(x). \end{aligned}$$

(v) $A \in \mathbf{O}(n, \mathbf{R})$ implies $AB := AA^t = A^t A = I$, that is, for all $1 \le i, j \le n$,

$$\sum_{1 \le k \le n} a_{ik} b_{kj} = \sum_{1 \le k \le n} a_{ik} a_{jk} = \delta_{ij}.$$

Hence the desired conclusion follows from part (iv).