Solution of Exercise 3.11.

(i) In view of $x_1 > 0$ and $x_2 > 0$ we have

$$f(0;x) = x_1^2 > 0, \qquad f(1;x) = 1-x_1^2-x_2^2-1+x_1^2 = -x_2^2 < 0, \qquad f(x_1^2+x_2^2+1;x) = x_1^2 > 0.$$

Application of the Intermediate Value Theorem 1.9.5 to the intervals $[\,0,1\,]$ and $[\,1, 1+x_1^2+x_2^2\,]$ now gives the existence of $y_1(x)$ and $y_2(x)$ such that

$$f(y_1(x);x) = 0 \quad \text{with} \quad 0 < y_1(x) < 1; \qquad f(y_2(x);x) = 0 \quad \text{with} \quad 1 < y_2(x).$$

(ii) On the basis of the definitions Φ is a mapping from U to V. For proving the surjectivity of Φ we show that given $y = (y_1, y_2) \in V$ we can find $x \in U$ such that y_1 and y_2 are the roots of the quadratic polynomial function $f(y;x)$. In fact, y_1 and y_2 are roots of the monic polynomial

$$(y-y_1)(y-y_2) = y^2 - (y_1+y_2)y + y_1y_2.$$

Comparison of coefficients now yields

$$y_1 + y_2 = x_1^2 + x_2^2 + 1, \qquad y_1y_2 = x_1^2.$$

In other words,

$$x_2^2 = y_1 + y_2 - x_1^2 - 1 = -y_1y_2 + y_1 + y_2 - 1 = (1-y_1)(y_2-1).$$

Hence there is a solution $x \in U$, given by

$$x_1 = \sqrt{y_1y_2}, \qquad x_2 = \sqrt{(1-y_1)(y_2-1)};$$

and moreover, this is the only possible one. Therefore, $\Phi : U \to V$ is bijective with the given $\Psi : V \to U$ as its inverse.

(iii) $\sqrt{} : [\,0,\infty\,[\, \to \mathbf{R}$ is continuous, and a C^∞ function on the open interval $]\,0,\infty\,[$. Since $y_1y_2 > 0$ and $(1-y_1)(y_2-1) > 0$ for $y \in V$, the mapping Ψ is of class C^∞. We have

$$\Psi(y) = \begin{pmatrix} \sqrt{y_1y_2} \\ \sqrt{(1-y_1)(y_2-1)} \end{pmatrix}, \qquad \text{so} \qquad 2D\Psi(y) = \begin{pmatrix} \sqrt{\frac{y_2}{y_1}} & \sqrt{\frac{y_1}{y_2}} \\ -\sqrt{\frac{y_2-1}{1-y_1}} & \sqrt{\frac{1-y_1}{y_2-1}} \end{pmatrix}.$$

Hence

$$\begin{aligned} 4\det D\Psi(y) &= \sqrt{\frac{y_2(1-y_1)}{y_1(y_2-1)}} + \sqrt{\frac{y_1(y_2-1)}{y_2(1-y_1)}} = \frac{y_2(1-y_1) + y_1(y_2-1)}{\sqrt{y_1y_2(1-y_1)(y_2-1)}} \\ &= \frac{y_2 - y_2y_1 + y_1y_2 - y_1}{\sqrt{y_1y_2}\sqrt{(1-y_1)(y_2-1)}} = \frac{y_2-y_1}{x_1x_2}. \end{aligned}$$

This implies $\det D\Psi(y) > 0$, and on the basis of the Global Inverse Function Theorem 3.2.8 we may conclude that $\Psi : V \to U$ is a C^∞ diffeomorphism. But then $\Phi : U \to V$ is one too.

(iv) We have $y(y-1)g_y(x) = -f(y;x)$, while $y(y-1) \neq 0$ for $y \in I$.

(v) According to part (ii), for every $x \in U$, there exists unique $y \in V$ such that

$$0 < y_1 < 1, \qquad g_{y_1}(x) = 0 \qquad \text{and} \qquad 1 < y_2, \qquad g_{y_2}(x) = 0.$$

In other words,

$$x \in V_{y_1} \quad \text{with } V_{y_1} \text{ hyperbola} \qquad \text{and} \qquad x \in V_{y_2} \quad \text{with } V_{y_2} \text{ ellipse.}$$

Moreover, it follows from (ii) that given $y \in V$ there is unique $x = \Psi(y) \in U$ such that $g_{y_1}(x) = g_{y_2}(x) = 0$. That is,

$$V_{y_1} \cap V_{y_2} \cap U = \{x\}.$$