Solution of Exercise 6.15.

(i) We may assume $a > 0$. First note that by symmetry of the integrand

$$\Big(\int_0^a e^{-x^2}\,dx\Big)^2 = \int_K e^{-\|x\|^2}\,dx = 2\int_L e^{-\|x\|^2}\,dx,$$

where

$$K = \{\, x \in \mathbf{R}^2 \mid 0 \le x_j \le a,\ 1 \le j \le 2\,\} \qquad \text{and} \qquad L = \{\, x \in K \mid x_2 \le x_1\,\}.$$

Introducing polar coordinates (r, α) in $\mathbf{R}^2$ as in Example 6.6.4, we see that $0 \le x_1 = r\cos\alpha \le a$ implies $0 \le r \le \frac{a}{\cos\alpha}$; thus

$$L = \{\, r(\cos\alpha, \sin\alpha) \in \mathbf{R}^2 \mid 0 \le \alpha \le \frac{\pi}{4},\ 0 \le r \le \frac{a}{\cos\alpha}\,\}.$$

As in the example we obtain

$$\Big(\int_0^a e^{-x^2}\,dx\Big)^2 = \int_0^{\frac{\pi}{4}}\int_0^{\frac{a}{\cos\alpha}} e^{-r^2} 2r\,dr\,d\alpha = \int_0^{\frac{\pi}{4}} \Big[-e^{-r^2}\Big]_0^{\frac{a}{\cos\alpha}}\,d\alpha = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} e^{-\frac{a^2}{\cos^2\alpha}}\,d\alpha.$$

(ii) If $\alpha = \arctan t$, then $\tan\alpha = t$; hence, $0 \le \alpha \le \frac{\pi}{4}$ implies $0 \le t \le 1$, while

$$\frac{1}{\cos^2\alpha} = \frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha} = 1 + \tan^2\alpha = 1 + t^2 \qquad \text{and} \qquad d\alpha = \frac{1}{1+t^2}\,dt.$$

As a consequence

$$\int_0^{\frac{\pi}{4}} e^{-\frac{a^2}{\cos^2\alpha}}\,d\alpha = \int_0^1 \frac{e^{-a^2(1+t^2)}}{1+t^2}\,dt.$$