Solution of Exercise 6.20. Consider arbitrary x and $y \in \mathbf{R}^3$ and $A \in \mathbf{SO}(3, \mathbf{R})$. Then $\langle x, A^{-1}y\rangle = \langle Ax, AA^{-1}y\rangle = \langle Ax, y\rangle$, while $U = AU$ and $\det DA(x) = \det A = 1$. Accordingly

$$\int_U \cos\langle x, A^{-1}y\rangle\, dx = \int_{AU} \cos\langle Ax, y\rangle \, \det DA(x)\, dx = \int_U \cos\langle x, y\rangle\, dx.$$

Here, in the latter equality, the Change of Variables Theorem 6.6.1 has been used. In view of the equality above, given any $y \in \mathbf{R}^3$, we may assume $y = \|y\|(0, 0, 1)$; this implies $\langle x, y\rangle = x_3\|y\|$. Therefore we have to evaluate $\int_U \cos(\|y\|x_3)\, dx$. To this end, substitute spherical coordinates as in Exercise 3.18, in particular, $x_3 = r \sin\theta$ and $dx = r^2 \cos\theta\, dr\, d\alpha\, d\theta$. On the basis of Corollary 6.4.3 on interchanging the order of integration we obtain, for $y \in \mathbf{R}^3 \setminus \{0\}$,

$$\int_0^1 \int_\pi^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(r\|y\| \sin\theta) r^2 \cos\theta\, dr\, d\alpha\, d\theta = \frac{2\pi}{\|y\|} \int_0^1 r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(r\|y\| \sin\theta) r\|y\| \cos\theta\, d\theta\, dr$$

$$= \frac{2\pi}{\|y\|} \int_0^1 r \big[\sin(r\|y\| \sin\theta)\big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\, dr = \frac{4\pi}{\|y\|} \int_0^1 r \sin(r\|y\|)\, dr.$$

Integration by parts now implies

$$\int_0^1 r \sin(r\|y\|)\, dr = \Big[-\frac{r\cos(r\|y\|)}{\|y\|}\Big]_0^1 + \int_0^1 \frac{\cos(r\|y\|)}{\|y\|}\, dr = -\frac{\cos\|y\|}{\|y\|} + \Big[\frac{\sin(r\|y\|)}{\|y\|^2}\Big]_0^1$$

$$= \frac{\sin\|y\|}{\|y\|^2} - \frac{\cos\|y\|}{\|y\|}.$$

Next, Taylor expansion of the functions $\sin$ and $\cos$ about 0 gives

$$\frac{4\pi}{\|y\|^2}\Big(\frac{\sin\|y\|}{\|y\|} - \cos\|y\|\Big) = \frac{4\pi}{\|y\|^2}\Big(1 - \frac{\|y\|^2}{6} - 1 + \frac{\|y\|^2}{2} + \mathcal{O}(\|y\|^3)\Big) = \frac{4}{3}\pi(1 + \mathcal{O}(\|y\|)), \quad \|y\| \to 0.$$

Thus, by taking the limit for $\|y\| \to 0$ and applying the Continuity Theorem 2.10.2, we get the formula for $\mathrm{vol}_3(U)$.