Solution of Exercise 6.23.

(i) This follows directly from Exercise 1.15 and the observation that the mappings Ψ and Φ are of class C^∞.

(ii) The equality

$$\Psi_i(y) = \Big(1 - \sum_{1\le k\le n} y_k^2\Big)^{-\frac{1}{2}} y_i$$

implies

$$D_j\Psi_i(y) = y_j\Big(1 - \sum_{1\le k\le n} y_k^2\Big)^{-\frac{3}{2}} y_i + \Big(1 - \sum_{1\le k\le n} y_k^2\Big)^{-\frac{1}{2}} \delta_{ij} = (1-\|y\|^2)^{-\frac{1}{2}}\,(\Psi_i(y)\Psi_j(y) + \delta_{ij}).$$

(iii) On the basis of the multiplicative properties of the determinant and the fact that $\det AA^t = 1$, which is valid for $A \in \mathbf{O}(n,\mathbf{R})$, we see

$$\begin{aligned}\det(I + xx^t) &= \det A \det(I + xx^t)\det A^t = \det(AA^t + Axx^tA^t) = \det\big(I + (Ax)(Ax)^t\big)\\ &= \det(I + zz^t).\end{aligned}$$

In particular, we may select $A \in \mathbf{O}(n,\mathbf{R})$ such that $z = Ax = \|x\|e_1$, where e_1 is the first standard basis vector in $\mathbf{R}^n$. Then

$$I + zz^t = \begin{pmatrix} 1+\|x\|^2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Combination of the two equalities above now leads to the desired result.

(iv) Application of the parts (ii) and (iii) gives

$$\begin{aligned}\det D\Psi(y) &= (1-\|y\|^2)^{-\frac{n}{2}}\,\det\big(I + \Psi(y)\Psi(y)^t\big) = (1-\|y\|^2)^{-\frac{n}{2}}\Big(1 + \frac{\|(y)\|^2}{1-\|y\|^2}\Big)\\ &= \frac{1}{(1-\|y\|^2)^{\frac{n}{2}+1}}.\end{aligned}$$

Note that $(\star)$ in the solution to Exercise 1.15, with f replaced by Φ and the roles of x and y reversed, implies

$$1 + \|x\|^2 = 1 + \frac{\|y\|^2}{1-\|y\|^2} = \frac{1}{1-\|y\|^2}$$

if $x = \Psi(y)$; and according to Example 2.4.9 this gives

$$\det D\Phi(x) = \frac{1}{\det D\Psi(y)} = (1-\|y\|^2)^{\frac{n}{2}+1} = \frac{1}{(1+\|x\|^2)^{\frac{n}{2}+1}}.$$

The identities for the integrals now follow by the Change of Variables Theorem 6.6.1.