Solution of Exercise 6.50.

(i) Integrating by parts one obtains, since $p \in \mathbf{R}_+$,

$$\Gamma(p+1) = \int_{\mathbf{R}_+} e^{-t}\, t^p\, dt = [-e^{-t}t^p]_0^\infty + p\int_{\mathbf{R}_+} e^{-t}\, t^{p-1}\, dt = p\,\Gamma(p).$$

The identity $\Gamma(n+1) = n!$ now follows by mathematical induction as $\Gamma(1) = \int_{\mathbf{R}_+} e^{-t}\, dt = 1$. Setting $t = u^2$ one finds

$$(\star) \qquad \Gamma(p) = 2\int_{\mathbf{R}_+} e^{-u^2} u^{2p-2+1}\, du = 2\int_{\mathbf{R}_+} e^{-u^2} u^{2p-1}\, du.$$

Therefore one deduces from Example 6.10.8

$$\Gamma\Big(\frac{1}{2}\Big) = 2\int_{\mathbf{R}_+} e^{-u^2}\, du = \sqrt{\pi}.$$

(ii) Successively using $(\star)$ above, Corollary 6.4.3 on writing an iterated integral as a two-dimensional integral, polar coordinates $u = r(\cos\alpha, \sin\alpha)$ in $\mathbf{R}_+^2$ as in Example 6.6.4 and $(\star)$ once more, one sees

$$\begin{aligned}
\Gamma(p_1)\Gamma(p_2) &= 2\int_{\mathbf{R}_+} e^{-u_1^2} u_1^{2p_1-1}\, du_1\, 2\int_{\mathbf{R}_+} e^{-u_2^2} u_2^{2p_2-1}\, du_2 \\
&= 4\int_{\mathbf{R}_+^2} e^{-(u_1^2+u_2^2)} u_1^{2p_1-1} u_2^{2p_2-1}\, du \\
&= 4\int_{\mathbf{R}_+} e^{-r^2} r^{2(p_1+p_2)-2+1}\, dr \int_0^{\frac{\pi}{2}} \cos^{2p_1-1}\alpha \sin^{2p_2-1}\alpha\, d\alpha \\
&= 2\Gamma(p_1+p_2)\int_0^{\frac{\pi}{2}} \cos^{2p_1-1}\alpha \sin^{2p_2-1}\alpha\, d\alpha.
\end{aligned}$$

Accordingly

$$\mathrm{B}(p_1, p_2) = \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1+p_2)} = 2\int_0^{\frac{\pi}{2}} \cos^{2p_1-1}\alpha \sin^{2p_2-1}\alpha\, d\alpha.$$

(iii) This is a straightforward consequence of part (ii) by a substitution of parameters.

(iv) Set $\cos^2\alpha = t$, then $-2\cos\alpha\sin\alpha\, d\alpha = dt$ and $\sin^2\alpha = 1-t$. Hence

$$\begin{aligned}
\mathrm{B}(p_1, p_2) &= -\int_0^{\frac{\pi}{2}} \cos^{2(p_1-1)}\alpha \sin^{2(p_2-1)}\alpha\, d\alpha = -\int_1^0 t^{p_1-1}(1-t)^{p_2-1}\, dt \\
&= \int_0^1 t^{p_1-1}(1-t)^{p_2-1}\, dt.
\end{aligned}$$

The mapping $\Psi : u \mapsto \frac{u}{u+1} = t$ is a C^∞ diffeomorphism $\mathbf{R}_+ \to\ ]\,0, 1\,[$ satisfying $1 - t = \frac{1}{u+1}$ and $\Psi'(u) = \frac{1}{(u+1)^2}$. On the basis of the Change of Variables Theorem 6.6.1 one therefore gets

$$\int_0^1 t^{p_1-1}(1-t)^{p_2-1}\, dt = \int_{\mathbf{R}_+} \frac{u^{p_1-1}}{(1+u)^{p_1+p_2}}\, du = 2\int_{\mathbf{R}_+} \frac{v^{2p_1-1}}{(1+v^2)^{p_1+p_2}}\, dv,$$

where the last equality is obtained by means of the substitution $u = v^2$.