Solution of Exercise 7.21.

(i) Use arguments similar to those in Example 6.10.8.

(ii) In Cartesian coordinates we have

$$\int_{\mathbf{R}^n} e^{-\|x\|^2}\,dx = \Big(\int_{\mathbf{R}} e^{-x^2}\,dx\Big)^n = \pi^{\frac{n}{2}},$$

on the basis of Example 6.10.8. On the other hand, application of Formula (7.26) and Exercise 6.50.(i) gives

$$\begin{aligned}\int_{\mathbf{R}^n} e^{-\|x\|^2}\,dx &= \int_{\mathbf{R}_+} r^{n-1}e^{-r^2}\int_{S^{n-1}} d_{n-1}y\,dr = \text{hyperarea}_{n-1}(S^{n-1})\int_{\mathbf{R}_+} e^{-r^2}r^{n-1}\,dr\\ &= \text{hyperarea}_{n-1}(S^{n-1})\frac{1}{2}\Gamma\Big(\frac{n}{2}\Big).\end{aligned}$$

Comparison of the two equations now leads to the desired equality.

(iii) Apply the formula from part (ii) for $n = 3$ and note that $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\pi^{\frac{1}{2}}}{2}$, according to Exercise 6.50.(i).

(iv) We have

$$\text{vol}_n(B^n) = \int_0^1 \text{hyperarea}_{n-1}(S^{n-1})r^{n-1}\,dr = \frac{1}{n}\,\text{hyperarea}_{n-1}(S^{n-1}).$$

(v) Combination of parts (iv) and (ii) as well as Exercise 6.50.(i) implies

$$\text{vol}_n(B^n) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

(vi) This follows by the same arguments as in part (v).