Solution of Exercise 7.44.

(i) On the basis of

$$\frac{\partial}{\partial x_1}\frac{x_1}{x_1^2+x_2^2}=\frac{1}{x_1^2+x_2^2}-\frac{x_1\,2x_1}{(x_1^2+x_2^2)^2}=\frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2},\qquad \frac{\partial}{\partial x_2}\frac{x_2}{x_1^2+x_2^2}=\frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}$$

we see $\operatorname{div} f(x)=0$. Gauss' Divergence Theorem 7.8.5 then implies

$$0=\int_\Omega \operatorname{div} f(x)\,dx=\int_{\partial\Omega}\langle f,\nu\rangle(y)\,d_2y.$$

The outer normal $\nu(y)$ to the two plane regions is $(0,0,\mp1)$, respectively, and the inner product of this vector with the vector $f(y)$ equals 0 for points y belonging to the two plane regions. For $y\in S_1$ we have $\nu(y)=-y$ and therefore

$$\langle f(y),\nu(y)\rangle=-\frac{1}{y_1^2+y_2^2}\langle (y_1,y_2,0),(y_1,y_2,y_3)\rangle=-1.$$

A parametrization of S_2 is given by

$$\phi:\,\Big]-\frac{\pi}{2},\frac{\pi}{2}\Big[\times\,]\,h_1,h_2\,[\,\to\mathbf{R}^3\qquad\text{with}\qquad \phi(\alpha,t)=(\cos\alpha,\sin\alpha,t).$$

Accordingly

$$\frac{\partial\phi}{\partial\alpha}\times\frac{\partial\phi}{\partial t}(\alpha,t)=(\cos\alpha,\sin\alpha,0),$$

which implies, for $y\in S_2$,

$$\nu(y)=(y_1,y_2,0),\qquad\text{hence}\qquad \langle f(y),\nu(y)\rangle=1.$$

Accordingly

$$0=-\int_{S_1}d_2y+\int_{S_2}d_2y.$$

(ii) Recognizing the shell S_1 as the difference of two caps of the sphere we deduce from Example 7.4.6

$$\operatorname{area}(S_1)=2\pi(1-\sin\arcsin h_1)-2\pi(1-\sin\arcsin h_2)=2\pi(h_2-h_1).$$

From the calculation in part (i) we obtain

$$\left\|\frac{\partial\phi}{\partial\alpha}\times\frac{\partial\phi}{\partial t}(\alpha,t)\right\|=\|(\cos\alpha,\sin\alpha,0)\|=1,$$

which implies

$$\operatorname{area}(S_2)=\int_{-\pi}^{\pi}\int_{h_1}^{h_2}d\alpha\,dt=2\pi(h_2-h_1).$$