Solution of Exercise 7.46. From Euler's identity in Exercise 2.32.(ii) one obtains, for $y \in S^{n-1}$,

$$\langle \operatorname{grad} f(y), y \rangle = d\, f(y).$$

Furthermore $\Delta f = \operatorname{div}(\operatorname{grad} f)$ and $\nu(y) = y$, for $y \in S^{n-1}$, therefore Gauss' Divergence Theorem 7.8.5 implies

$$\int_{B^n} \Delta f(x)\, dx = \int_{S^{n-1}} \langle \operatorname{grad} f(y), y \rangle \, d_{n-1}y = d \int_{S^{n-1}} f(y)\, d_{n-1}y.$$

Finally, the identity $\Delta(x \mapsto x_j^2) = 2$ leads to

$$2 \operatorname{vol}_n(B^n) = 2 \int_{S^{n-1}} y_j^2 \, d_{n-1}y.$$