Solution of Exercise 8.5.

(i) The vectors $\phi(t_+) - \phi(t_-)$ and $\phi(t_+) - \phi(t) \in \mathbf{R}^2$ span a parallelogram of twice the area of that of the triangle; hence, the latter area equals

$$\frac{1}{2}\det\left(\phi(t_+) - \phi(t_-)\ \ \phi(t_+) - \phi(t)\right) = \frac{1}{2}\begin{vmatrix} t_+ - t_- & t_+ - t \\ t_+^2 - t_-^2 & t_+^2 - t^2 \end{vmatrix}$$

$$= \frac{(t_+ - t_-)(t_+ - t)}{2}\begin{vmatrix} 1 & 1 \\ t_+ + t_- & t_+ + t \end{vmatrix} = \frac{(t_+ - t_-)(t_+ - t)(t - t_-)}{2}.$$

This quadratic function in t attains its maximum at $t = \frac{t_+ + t_-}{2} = t_0$.

(ii) The direction of the tangent line to P at $\phi(t)$ is given by $\phi'(t) = (1, 2t)$ and therefore the slope of the tangent equals $2t$. The slope of $l(t_+, t_-)$ is $\frac{t_+^2 - t_-^2}{t_+ - t_-} = t_+ + t_-$. Thus, $t = t_0$.

(iii) We begin with the proof by successive integration. We have

$$l(t_+, t_-) = \{\, \phi(t_-) + t(\phi(t_+) - \phi(t_=)) \mid 0 < t < 1 \,\}.$$

Thus, $x \in l(t_+, t_-)$ if and only if there exists $t \in \mathbf{R}$ such that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t_- + t(t_+ - t_-) \\ t_-^2 + t(t_+^2 - t_-^2) \end{pmatrix}.$$

So $t(t_+ - t_-) = x_1 - t_-$, which implies

$$x_2 = t_-^2 + (x_1 - t_-)(t_+ + t_-) = (t_0 - \delta)^2 + 2t_0(x_1 - t_-).$$

Obviously, this leads to the desired description of $S(t_+, t_-)$, since points in $S(t_+, t_-)$ lie above P but below $l(t_+, t_-)$. Furthermore,

$$\begin{aligned}
\operatorname{area}(S(t_+, t_-)) &= \int_{t_-}^{t_+} \int_{x_1^2}^{(t_0-\delta)^2 + 2t_0(x_1 - t_-)} dx_2\, dx_1 \\
&= \int_{t_-}^{t_+} ((t_0 - \delta)^2 + 2t_0(x_1 - t_-) - x_1^2)\, dx_1 \\
&= (t_0 - \delta)^2(t_+ - t_-) + t_0(t_+^2 - t_-^2) - 2t_0 t_-(t_+ - t_-) - \tfrac{1}{3}(t_+^3 - t_-^3) \\
&= 2(t_0 - \delta)^2\delta + 4t_0^2\delta - 4t_0\delta(t_0 - \delta) - \frac{2}{3}\delta^3 - 2t_0^2\delta \\
&= 2t_0^2\delta - 4t_0\delta^2 + 2\delta^3 + 4t_0\delta^2 - \frac{2}{3}\delta^3 - 2t_0^2\delta = \frac{4\delta^3}{3}.
\end{aligned}$$

For the proof by means of Green's Integral Theorem 8.3.5 we note that the positive parametrization of the boundary of $S(t_+, t_-)$ consists of the following two pieces:

$$\partial_1 S(t_+, t_-) = \{\, \phi(t) \mid t_- < t < t_+ \,\},$$
$$\partial_2 S(t_+, t_-) = \{\, \phi(t_+) + t(\phi(t_-) - \phi(t_+)) \mid 0 < t < 1 \,\}.$$

In view of Formula (8.26) we compute for $\partial_1 S(t_+, t_-)$

$$(\phi_1\phi_2' - \phi_2\phi_1')(t) = t \cdot 2t - t^2 \cdot 1 = t^2,$$

while for $\partial_2 S(t_+, t_-)$ we have

$$\begin{aligned}(y_1y_2' - y_2y_1')(t) &= (t_+ + t(t_- - t_+))(t_-^2 - t_+^2) - (t_+^2 + t(t_-^2 - t_+^2))(t_- - t_+)\\ &= t_+(t_-^2 - t_+^2) - t_+^2(t_- - t_+) = (t_- - t_+)(t_+t_- + t_+^2 - t_+^2)\\ &= t_+t_-(t_- - t_+).\end{aligned}$$

Accordingly

$$\begin{aligned}\text{area}\,(S(t_+, t_-)) &= \frac{1}{2}\int_{t_-}^{t_+} t^2\,dt + \frac{1}{2}\int_0^1 t_+t_-(t_- - t_+)\,dt = \frac{1}{6}(t_+^3 - t_-^3) - t_+t_-\delta\\ &= \frac{\delta^3}{3} + t_0^2\delta - t_+t_-\delta = \frac{\delta^3}{3} + \frac{\delta}{4}(t_+^2 + t_-^2 + 2t_+t_- - 4t_+t_-)\\ &= \frac{\delta^3}{3} + \delta\frac{(t_+ - t_-)^2}{4} = \frac{4\delta^3}{3}.\end{aligned}$$

The third proof is by recognizing $S(t_+, t_-)$ as the set-theoretical difference of the trapezoid with vertices $(t_-, 0)$, $\phi(t_-)$, $\phi(t_+)$ and $(t_+, 0)$ and the graph of ϕ above $[t_-, t_+]$. This leads to

$$\begin{aligned}\text{area}\,(S(t_+, t_-)) &= \frac{1}{2}(t_-^2 + t_+^2)(t_+ - t_-) - \int_{t_-}^{t_+} t^2\,dt = \frac{1}{2}(t_-^2 + t_+^2)(t_+ - t_-) - \frac{1}{3}(t_+^3 - t_-^3)\\ &= \frac{1}{6}(t_+ - t_-)(3t_-^2 + 3t_+^2 - 2t_+^2 - 2t_+t_- - 2t_-^2) = \frac{1}{6}(t_+ - t_-)^3 = \frac{4\delta^3}{3}.\end{aligned}$$

(iv) The endomorphism of $\mathbf{R}^2$ with matrix $\delta\begin{pmatrix} 1 & 0 \\ 2t_0 & \delta \end{pmatrix}$ is invertible, having determinant equal to $\delta^3 > 0$. Therefore Ψ is an invertible affine transformation and consequently a C^∞ diffeomorphism of $\mathbf{R}^2$. We have

$$\Psi(\phi(t)) = \begin{pmatrix} t_0 + \quad \delta\,t \\ t_0^2 + 2t_0\delta\,t + \delta^2t^2 \end{pmatrix} = \phi(t_0 + \delta\,t).$$

Since a point belongs to P if and only if it is of the form $\phi(t)$, for some $t \in \mathbf{R}$, it follows that Ψ maps P into itself. Furthermore

$$\Psi(\ \ 1, 1) = \Psi \circ \phi(\ \ 1) = \phi(t_0 + \delta) = \phi(t_+), \qquad \Psi(0, 0) = \Psi \circ \phi(0) = \phi(t_0),$$
$$\Psi(-1, 1) = \Psi \circ \phi(-1) = \phi(t_0 - \delta) = \phi(t_-).$$

Ψ being an affine mapping, it now follows that Ψ maps the triangle $\Delta(1, -1)$ onto the triangle $\Delta(t_+, t_-)$ and that, in addition, it maps the sector $S(1, -1)$ onto the sector $S(t_+, t_-)$.

(v) In part (iv) it has been proved that Ψ has constant Jacobi determinant equal to δ^3. The rectangle with vertices $(-1, 0)$, $\phi(-1)$, $\phi(1)$ and $(1, 0)$ has area 2 and the graph of ϕ above $[-1, 1]$ has area $\int_{-1}^1 t^2\,dt = \frac{2}{3}$; hence, $S(1, -1)$ has area $\frac{4}{3}$ while $\Delta(1, -1)$ has area 1. That shows that the formula for the quadrature of the parabola is valid in this case. Furthermore, Ψ maps this special configuration onto the general configuration under multiplication of areas with the same factor, which implies the quadrature of the parabola in the general case.