Multidimensional Real Analysis
Corrigenda and Addenda

Mathematical mistakes are indicated by the symbol ▼; the majority of the corrections are minor textual changes.

Example 1.3.10 on page 16. *Insert the following words immediately after* **Example 1.3.10***:* **(Plücker's conoid).**

Lemma 2.1.1 on page 40. *There exists a slightly less computational proof of this lemma.*
In fact, apply Lemma 1.1.7.(ii) to $Ah = \sum_{1\le j\le n} h_j Ae_j$, which follows from Formula (1.1), in order to obtain

$$\|Ah\| \le \sum_{1\le j\le n} |h_j|\,\|Ae_j\| = \langle\,(|h_1|,\dots,|h_n|),\ (\|Ae_1\|,\dots,\|Ae_n\|)\,\rangle.$$

Now the Cauchy–Schwarz inequality from Proposition 1.1.6 immediately yields

$$\|Ah\| \le \|h\|\sqrt{\sum_{1\le j\le n}\|Ae_j\|^2} = \|h\|\,\|A\|_{\text{Eucl}}.$$

Lemma 2.2.7 on page 45. *As additional motivation for the proof of Hadamard's Lemma one may offer the following argument.*
On the one hand, for differentiable f, one requires

$$f(x) - f(a) = \phi_a(x)(x-a).$$

On the other hand, in view of $\|x-a\|^2 = (x-a)^t(x-a) \in \mathbf{R}$, the reformulation of differentiability in Formula (2.10) implies

$$\begin{aligned} f(x) - f(a) &= Df(a)(x-a) + \epsilon_a(x-a) \\ &= Df(a)(x-a) + \frac{1}{\|x-a\|^2}\epsilon_a(x-a)(x-a)^t(x-a). \end{aligned}$$

A formal division of the right–hand side by $x-a$ now suggests the formula for $\phi_a(x)$ as given in the proof.

Theorem 2.4.1 on page 51. *Replace the last sentence in the assertion of the chain rule by the following:*
And, if f is differentiable on U with $f(U) \subset V$ and g is differentiable on V,

$$D(g\circ f) = ((Dg)\circ f)\circ Df : U \to \operatorname{Lin}(\mathbf{R}^n, \mathbf{R}^q).$$

Proof of Corollary 2.4.3 on page 53. *Replace the first sentence by the following:*
From Example 2.2.5 we know that $f : \mathbf{R}^n \to \mathbf{R}^p \times \mathbf{R}^p$ is differentiable at a if $f(x) = (f_1(x), f_2(x))$, while $Df(a)h = (Df_1(a)h, Df_2(a)h)$, for $(a, h \in \mathbf{R}^n)$.

Definition 3.1.2 on page 88. *Replace the displayed formula by the following:*

$$\Psi^* f = f\circ\Psi : V \to \mathbf{R}^p, \qquad \text{that is} \qquad \Psi^* f(y) = f(\Psi(y)) \qquad (y \in V),$$

Subsection 3.4.(C) on page 98. *Replace the first display in this subsection by the following:*

$$\mathbf{R}^p \rightarrowtail \mathbf{R}^n \times \mathbf{R}^p \rightarrowtail \mathbf{R}^n \qquad \text{given by} \qquad y \overset{\psi\times I}{\longmapsto} (\psi(y), y) \overset{f}{\mapsto} f(\psi(y), y) = 0.$$

Theorem 3.5.1 on page 100. *Replace the sentence following the first display in the theorem by the following:*
Then there exist open neighborhoods U of x^0 in $\mathbf{R}^n$ and V of y^0 in $\mathbf{R}^p$ with the following properties: $U \times V \subset W$ and

Application A on page 101. *Replace the last sentence (on page 102) in the application by the following:*
That theorem in algebra asserts that there does not exist a formula which gives the zeros of a general polynomial function of degree n in terms of the coefficients of that function by means of addition, subtraction, multiplication, division and extraction of roots, if $n \geq 5$ and one even works over $\mathbf{C}$.

Application B on page 103. *Replace the second sentence by the following:*
Consider the following equation for $x \in \mathbf{R}$ with $y \in \mathbf{R}$ as a parameter

Application D on page 104. *Replace the sentence following the second display by the following:*
Now, for all $(x; y) \in \mathbf{R}^n \times \mathbf{R}^{n^2+n}$ and $1 \leq i, j \leq n$,

Text preceding Definition 4.1.2 on page 108. *Replace the last sentence preceding the definition by the following:*
There are two other common ways of specifying sets V: in Definitions 4.1.2 and 4.1.3 the set V is described as an **image**, or **inverse image**, respectively, under a mapping.

Proof of Rank Lemma 4.2.7 on page 113. *Replace the first part of the proof by the following:*
Only (i) $\Rightarrow$ (ii) needs verification. In view of the equality $\dim(\ker A) + r = n$, we can find a basis $(a_{r+1}, \dots, a_n)$ of $\ker A \subset \mathbf{R}^n$ and vectors $a_1, \dots, a_r$ complementing this basis to a basis of $\mathbf{R}^n$. Define $\Psi \in \mathrm{Aut}(\mathbf{R}^n)$ setting $\Psi e_j = a_j$, for $1 \leq j \leq n$. Then $(A\Psi)e_j = Aa_j$, for $1 \leq j \leq r$, and $(A\Psi)e_j = 0$, for $r < j \leq n$. The vectors $b_j = Aa_j$, for $1 \leq j \leq r$, form a basis of $\mathrm{im}\, A \subset \mathbf{R}^p$. Let us complement them by vectors $b_{r+1}, \dots, b_p$ to a basis of $\mathbf{R}^p$. Define $\Phi \in \mathrm{Aut}(\mathbf{R}^p)$ by $\Phi b_i = e'_i$, for $1 \leq i \leq p$. Then the operators Φ and Ψ are the required ones, since

$$(\Phi \circ A \circ \Psi)e_j = \begin{cases} e'_j, & 1 \leq j \leq r; \\ 0, & r < j \leq n. \end{cases}$$

Proof of Rank Lemma 4.2.7 on page 114. *Insert the following immediately after the proof:*
Alternatively, the equality of the ranks of A and A^t may be verified as follows. We have $\mathbf{R}^n = \ker A \oplus \mathrm{im}\, A^t$. In fact,

$$\begin{aligned} x \in \ker A \quad &\Longleftrightarrow \quad Ax = 0 \quad \Longleftrightarrow \quad \langle\, Ax, y \,\rangle = 0 \quad (y \in \mathbf{R}^p) \\ &\Longleftrightarrow \quad \langle\, x, A^t y \,\rangle = 0 \quad (y \in \mathbf{R}^p) \quad \Longleftrightarrow \quad x \in (\mathrm{im}\, A^t)^\perp. \end{aligned}$$

Hence $(\ker A)^\perp = (\mathrm{im}\, A^t)^{\perp\perp} = \mathrm{im}\, A^t$ and so the equality follows from $\mathbf{R}^n = \ker A \oplus (\ker A)^\perp$. Furthermore, we know $\dim \mathbf{R}^n = \dim \ker A + \mathrm{rank}\, A$.

Theorem 4.3.1 on page 114. *Replace the first sentence of the theorem by the following:*
Let $d < n$ and let $D_0 \subset \mathbf{R}^d$ be an open subset, let $k \in \mathbf{N}_\infty$ and let $\phi : D_0 \to \mathbf{R}^n$ be a C^k mapping.

Corollary 4.3.2 on page 116. *Replace the initial part of the first sentence of the corollary by the following:*
Let let $d < n$ and $D \subset \mathbf{R}^d$ be a nonempty open subset, suppose $k \in \mathbf{N}_\infty$,

Example 4.5.1 on page 121. *Replace the last sentence of the first paragraph by the following:*
Note that in these (y, c)-coordinates a circle is locally described as the **affine** submanifold of $\mathbf{R}^3$ given by c equals a constant vector.

Theorem 4.5.2 on page 121. *Replace assertion (i) of the theorem by the following:*
The restriction of g to U is an open surjection onto C.
Replace assertion (iii) of the theorem by the following:
There exists a C^k diffeomorphism $\Phi : U \to \Phi(U) \subset \mathbf{R}^n$ such that Φ maps the manifold $N(c) \cap U$ in $\mathbf{R}^n$ into the affine submanifold of $\mathbf{R}^n$ given by

$$\{\, (x_1, \dots, x_n) \in \mathbf{R}^n \mid (x_{d+1}, \dots, x_n) = c \,\}.$$

Remark on page 124. *Replace the last sentence by the following:*
The fibers $N(c)$ together form a *fiber bundle*: under the diffeomorphism Φ from the Submersion Theorem they are locally transferred into the affine submanifolds of $\mathbf{R}^n$ given by the last $n - d$ coordinates being constant.

Example 4.6.2 on page 124. *Add the following sentence at the end of the example:*
See Exercises 4.22 and 5.58 for an explicit description of $\mathbf{SO}(3, \mathbf{R})$ and Exercise 4.23 for more details on $\mathbf{SO}(n, \mathbf{R})$, the subgroup of $\mathbf{O}(n, \mathbf{R})$ consisting of matrices of determinant 1.

Theorem 4.7.1 on page 126. *Replace the display in assertion (iii) of the theorem by the following:*

$$V \cap U = N(g, 0) = \{\, x \in U \mid g(x) = 0 \,\}.$$

Replace the initial part of assertion (iv) of the theorem by the following:
There exist an open neighborhood U in $\mathbf{R}^n$ of x, a C^k diffeomorphism $\Phi : U \to \Phi(U)$ in $\mathbf{R}^n$ and an open subset Y of $\mathbf{R}^d$ such that

Remark on page 135. *Replace the first sentence by the following:*
In classical textbooks, and in drawings, it is more common to refer to the affine manifold $x + T_x V$ as the tangent space of V at the point x: the affine manifold which has a contact of order 1 with V at x.

Example 5.3.2 on page 138. *Add the following sentence at the end of the example:*
Therefore the angle itself always equals $\frac{\pi}{4}$.

Example 5.3.3 on page 138. *Insert the following words immediately after* **Example 5.3.3***:*
(Parametrized surface).

Example 5.3.4 on page 139. *Insert the following words immediately after* **Example 5.3.4***:*
(Space curve given by equations).

Example 5.3.4 on page 140. *Add the following at the end of the example:*
Phrased differently in terms of the cross product, we have

$$T_x V = \mathbf{R}(\operatorname{grad} g_1(x) \times \operatorname{grad} g_2(x)).$$

Example 5.3.5 on page 140. *Replace the second displayed formula of the example by the following:*

$$Dg(x)h = 0 \quad \Longleftrightarrow \quad \langle\, \mathrm{grad}\, g_1(x), h\rangle = \cdots = \langle\, \mathrm{grad}\, g_{n-d}(x), h\rangle = 0.$$

Add the following at the end of the example:
Equations for the geometric tangent space $x + T_xV$ are obtained as follows. Consider $h \in x + T_xV$, then $h = x + k$ where $k \in T_xV$. Thus, $k = h - x$ implies

$$0 = Dg(x)k = Dg(x)(h - x) = Dg(x)h - Dg(x)x.$$

In other words, $x + T_xV$ arises as the set of solutions $h \in \mathbf{R}^n$ of a system of $n - d$ inhomogeneous linear equations or, more precisely,

$$x + T_xV = \{\, h \in \mathbf{R}^n \mid Dg(x)h = Dg(x)x \,\}.$$

Example 5.3.8 on page 144. *Replace the two sentences preceding the first display in the example as well as the display by:*
Note that $\gamma'(t) = (2t,\, 3t^2) \in \mathrm{Lin}(\mathbf{R}, \mathbf{R}^2)$ is injective, unless $t = 0$. Furthermore, $\|\gamma'(t)\| = 2|t|\sqrt{1 + (\frac{3}{2}t)^2}$. For $t \neq 0$ we therefore have the normalized tangent vector

$$T(t) = \|\gamma'(t)\|^{-1}\gamma'(t) = \frac{\mathrm{sgn}\, t}{\sqrt{1 + (\frac{3}{2}t)^2}} \begin{pmatrix} 1 \\ \dfrac{3t}{2} \end{pmatrix}.$$

The superscript t *in the first formula for $\gamma'(t)$ indicates taking the transpose, but might cause confusion.*

Example 5.3.11 on page 145. *Replace the first sentence following the fourth display from below on page 146 by:*
Therefore, if $e_1, \ldots, e_{n-1}$ are the standard basis vectors of $\mathbf{R}^{n-1}$, it follows that T_xV is spanned by the vectors $u_j := (e_j,\, D_jh(x'))$, for $1 \leq j < n$.

Example 5.5.1 on page 151. *Replace the final part of the second and the third sentence by:*
here $a \in S$ and $c \in \mathbf{R}$ is nonnegative. (Verify that every affine submanifold of dimension $n - 1$ can be represented in this form.)

Example 5.5.1 on page 151. *Insert the following at the end of the example:*
Recall that Theorem 1.8.8 implies that the distance attains a minimal value.

Example 5.5.3 on page 152. *Insert the following at the end of the example on page 153:*
Geometrically, Hadamard's inequality follows from the following observations. The volume of the parallelepiped spanned by the vectors $a_1, \ldots, a_n$ does not change upon replacement of the vector a_n by its component p_n perpendicular to the hyperplane spanned by the vectors $a_1, \ldots, a_{n-1}$, because the determinant is a multilinear and antisymmetric function. Furthermore $\|p_n\| \leq \|a_n\|$ by Pythagoras' Theorem. Hence one obtains by downward mathematical induction on $n \geq j \geq 1$

$$|\det(a_1 \cdots a_n)| = \prod_{1 \leq j \leq n} \|p_j\| \leq \prod_{1 \leq j \leq n} \|a_j\|.$$

▼ **Exercise 0.3 on page 177.** *Replace the exercise by the following:*
We have

$$\arctan x + \arctan\frac{1}{x} = \pm\frac{\pi}{2} \qquad (x \gtrless 0).$$

Prove this by means of the following three methods.

(i) Set $\arctan x = \alpha$ and express $\frac{1}{x}$ in terms of α.

(ii) Use differentiation.

(iii) Recall that $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$, and make a substitution of variables.

Deduce $\lim_{x\to\infty} x(\frac{\pi}{2} - \arctan x) = 1$.

(iv) More generally show, for all x and $y \in \mathbf{R}$,

$$\arctan x + \arctan y = \begin{cases} \arctan\Big(\dfrac{x+y}{1-xy}\Big), & xy < 1; \\ \pm\dfrac{\pi}{2}, & xy = 1,\ y \gtrless 0; \\ \arctan\Big(\dfrac{x+y}{1-xy}\Big) \pm \pi, & xy > 1,\ y \gtrless 0. \end{cases}$$

Exercise 0.20 on page 191. *Replace the final part of the second sentence by the following:*

$$\zeta(2n) := \sum_{k\in\mathbf{N}} \frac{1}{k^{2n}} = (-1)^{n-1}\frac{1}{2}(2\pi)^{2n}\frac{B_{2n}}{(2n)!}.$$

▼ **Exercise 4.8.(ii) on page 296.** *Replace the assertion by the following:*
Prove that a point $x \in \mathbf{R}^3$ belongs to the helicoid if and only if $x_1 \sin\frac{x_3}{a} - x_2\cos\frac{x_3}{a} = 0$.

▼ **Exercise 5.18.(iii) on page 323.** *Replace the assertion by the following:*
Show that

$$L = \{\,(r,\alpha) \in [\,-1,1\,] \times \Big[\,-\frac{\pi}{4}, \frac{\pi}{4}\Big] \mid r^2 = \cos 2\alpha\,\}.$$

▼ **Exercise 5.51.(ii) on page 357.** *Replace the first sentence by the following:*
Using the substitution $t = \cos\alpha$, prove

▼ **Proof. of Theorem 6.2.8 on page 428.** *Replace the first display by the following:*

$$\sup_B f - \inf_B f = (\sup_B f_+ - \inf_B f_+) + (\sup_B f_- - \inf_B f_-);$$

Example 6.6.4 on page 447. *Replace the initial part of the sentence following the third display in the example by the following:*
Consider $-\pi \le \alpha_1 < \alpha_2 \le \pi$ and $\phi \in C(]\,\alpha_1, \alpha_2[)$ and suppose $\phi > 0$,

Example 6.6.8 on page 450. *Replace the fifth display in the example by the following:*

$$\det(x'(s)\ x'(s)) + \det(x(s)\ x''(s)) = \det(x(s)\ x''(s)) = 0.$$

Section 7.1 on page 487. *Replace the first sentence in the second paragraph of the section by the following:*
First we consider this problem locally, that is, in a sufficiently small neighborhood U in $\mathbf{R}^n$ of a point $x \in V$.

Example 7.4.1 on page 498. *Replace the last sentence in the first paragraph on page 499 by the following:*
Indeed, the ellipse is the image under the embedding $t \mapsto (a \sin t,\ b \cos t)$.

▼ **Footnote on page 504.** *The proof as given in the reference is erroneous, but other, correct, proofs do exist.*
For more details, see Casselman, B.: The difficulties of kissing in three dimensions. Notices Amer. Math. Soc. 51 (2004), 884-885.

Notation on page 512. *Replace the first sentence after the second display by the following:*
For $x = \phi(y) \in \partial\Omega \cap U$ the column vectors $D_j\phi(y)$ in the matrix $D\phi(y)$, for $1 \leq j < n$, form a basis for $T_x(\partial\Omega)$, the tangent space to $\partial\Omega$ at x.

▼ **Notation on page 512.** *Replace the initial part of the first sentence after the sixth display by the following:*
Note that $\partial\Omega \cap U \supset \Psi(\{0\} \times D)$;

Third display on page 523. *Replace the sentence containing this display by the following:*
Begin by assuming that S is a closed (and hence compact) subset of $\partial\Omega$ such that

$$\partial'\Omega := \partial\Omega \setminus S$$

is in fact an $(n-1)$-dimensional C^1 manifold, with Ω at one side of $\partial'\Omega$ at each point of $\partial'\Omega$.
Perform the substitution of W by $\partial'\Omega$ systematically up till Gauss' Divergence Theorem 7.8.5. In particular, replace Formula (7.54) by the following:

$$\int_\Omega D_j((1-\chi)\, f)(x)\, dx = \int_{\partial'\Omega} ((1-\chi)\, f\, \nu_j)(y)\, d_{n-1}y.$$

Replace the first sentence on page 524 by the following:
As a result, the left–hand side in (7.54) converges to $\int_\Omega (D_j f)(x)\, dx$, if $\epsilon \downarrow 0$; and the right–hand side in (7.54) converges to $\int_{\partial'\Omega} (f\, \nu_j)(y)\, d_{n-1}y$, if U shrinks to S.
Replace the last sentence of Gauss' Divergence Theorem 7.8.5 on page 529 by the following:
Then

$$\int_\Omega \operatorname{div} f(x)\, dx = \int \langle f, \nu \rangle (y)\, d_{n-1}y,$$

where the integration on the right–hand side is performed over $\partial\Omega$ or $\partial'\Omega$, respectively.

▼ **Exercise 6.9 on page 600.** *Replace the second sentence by the following:*
Prove $\int_B \|x\|^{-1}\, dx = 8 \log(1 + \sqrt{2})$.

Exercise 6.20 on page 602. *Replace the second sentence by the following:*
Prove, for all $y \in \mathbf{R}^3 \setminus \{0\}$,

Exercise 6.39 on page 612. *Add to the* **Background** *on page 613 the following:*
Using this functional equation one sees at once

$$\sum_{n\in\mathbf{N}} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{\log^2 2}{2}.$$

Replace the sentence on page 614 preceding the second display by the following:
Furthermore, the *Clausen function* $\mathrm{Cl}_2 : \mathbf{R} \to \mathbf{R}$, and the *Lobachevsky function* $\text{Л} : \mathbf{R} \to \mathbf{R}$ are defined by (see Exercise 0.18.(i) and use termwise integration)

Exercise 6.96 on page 657. *Replace the second formula in the display in part (i) by the following:*
$\mu := \int_{\mathbf{R}} x f_{\alpha,\lambda}(x)\,dx = \frac{\alpha}{\lambda}$,

Exercise 6.102 on page 665. *Replace the final part of the first sentence by the following:*
– needed for Exercises 7.30 and 8.20

Exercise 7.46 on page 704. *Add the following sentence at the end:*
Deduce $\mathrm{hyperarea}_{n-1}(S^{n-1}) = n\,\mathrm{vol}_n(B^n)$ (compare with Example 7.9.1 and Exercises 7.21.(iv), 7.35.(iii) and 7.45.(ii)).

Exercise 7.53 on page 706. *Add the following sentence at the end of part (v):*
In doing so, assume a function having the mean value property to belong to $C^2(\Omega)$.

▼ **Exercise 8.7.(i) on page 731.** *Replace the assertion by the following:*
Prove

$$\int_C \langle\, f(s), d_1 s\,\rangle = 3 \int_{\{\, x\in\mathbf{R}^2 \mid \|x\|\le 1\,\}} \|x\|^2\,dx = -\frac{3\pi}{2}.$$