Exercise 0.1 (Lambert's cylindrical projection). In $\mathbf{R}^3$ consider the sphere S^2, the subset S of S^2, and the cylinder C^2, respectively given by

$$S^2 = \{ x \in \mathbf{R}^3 \mid \|x\| = 1 \}, \qquad S = S^2 \setminus \{\pm(0,0,1)\}, \qquad C^2 = \{ x \in \mathbf{R}^3 \mid x_1^2 + x_2^2 = 1,\ |x_3| < 1 \}.$$

Introduce *Lambert's cylindrical projection* $f : S \to C^2$ as follows. Given $x \in S$, denote by ℓ_x the unique line in $\mathbf{R}^3$ containing x that is parallel to the plane $\{ x \in \mathbf{R}^3 \mid x_3 = 0 \}$ and that intersects the x_3-axis. Next define $f(x)$ to be the point of intersection of ℓ_x with C^2 of shortest distance to x.

(i) Prove that the mapping f is a bijection that is given by

$$f(x) = \Big(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, x_3 \Big).$$

Give a formula for the inverse $f^{-1} : C^2 \to S$.

Let V be a submanifold in $\mathbf{R}^3$ of dimension 2 that is contained in S.

(ii) Verify that $f(V)$ is a submanifold in $\mathbf{R}^3$ of dimension 2 that is contained in C^2 and show that V and $f(V)$ are of equal area.

Define

$$\psi : \,]-\pi, \pi\,[\times]-1, 1\,[\to C^2 \qquad \text{by} \qquad \psi(\alpha, x_3) = (\cos\alpha, \sin\alpha, x_3).$$

(iii) Prove that ψ is an embedding having an open dense subset C of C^2 as its image.

Define the *unrolling*

$$g : C \to \,]-\pi, \pi\,[\times]-1, 1\,[\subset \mathbf{R}^2$$

to be the inverse of ψ.

(iv) Show that W and the unrolling $g(W)$ have equal area, for every submanifold W in $\mathbf{R}^3$ of dimension 2 that is contained in C.

(v) Now consider the special case of $V \subset S^2$ being a *spherical diangle* with angle α, that is, V is the subset of S^2 bounded by two half great circles in S^2 whose tangent vectors at a point of intersection include an angle α. On the basis of parts (ii) and (iv) show that the area of V equals 2α (compare with Exercise 7.13.(i)). Conclude that the area of S^2 is given by 4π.

Solution of Exercise 0.1. (i) Suppose $x \in S$, then $x_1^2 + x_2^2 = 1 - x_3^2 \neq 0$. Furthermore, $\ell_x = \{ (\lambda x_1, \lambda x_2, x_3) \mid \lambda \in \mathbf{R} \}$. Now

$$(\lambda x_1, \lambda x_2, x_3) \in C^2 \qquad \Longrightarrow \qquad \lambda^2 (x_1^2 + x_2^2) = 1 \qquad \Longrightarrow \qquad \lambda = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}}.$$

The point of intersection of ℓ_x with C^2 closest to x is obtained by taking the plus sign. This proves the formula for f. Furthermore, given arbitrary $y \in C^2$, an element $x \in S$ such that $f(x) = y$ has to satisfy

$$x_3 = y_3, \qquad \Longrightarrow \qquad \sqrt{x_1^2 + x_2^2} = \sqrt{1 - x_3^2} = \sqrt{1 - y_3^2}, \qquad \Longrightarrow \qquad x_j = y_j \sqrt{1 - y_3^2},$$

for $1 \le j \le 2$. Indeed, such an x belongs to S, in view of

$$\|x\|^2 = (y_1^2 + y_2^2)(1 - y_3^2) + y_3^2 = 1.$$

As a consequence, $x \in S$ exists and is uniquely determined. This establishes the bijectivity of f and also that

$$f^{-1}(y) = (y_1\sqrt{1 - y_3^2},\, y_2\sqrt{1 - y_3^2},\, y_3).$$

(ii) As is well-known, up to subsets of negligible area, two-dimensional submanifolds V contained in S are of the form $V = \phi(D)$, with $\phi : D \to S^2$ given by

$$D \subset \,]-\pi, \pi\,[\, \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\qquad \text{and} \qquad \phi(\alpha, \theta) = (\cos\alpha\cos\theta, \sin\alpha\cos\theta, \sin\theta).$$

Note that we may take D to be open and that ϕ is an embedding. As in Example 7.4.6 we see

$$\text{area}(V) = \int_D \cos\theta \, d\alpha d\theta.$$

On account of f and f^{-1} being a differentiable bijections (on suitable open subsets of $\mathbf{R}^3$) we see that $\widetilde{\phi} = f \circ \phi : D \to C^2$ is an embedding, which is given by

$$\widetilde{\phi}(\alpha, \theta) = (\cos\alpha, \sin\alpha, \sin\theta).$$

$f(V) = \widetilde{\phi}(D)$ is a submanifold in $\mathbf{R}^3$ of dimension 2 that is contained in C^2 because of Corollary 4.3.2. Furthermore,

$$\frac{\partial\widetilde{\phi}}{\partial\alpha}(\alpha, \theta) = (-\sin\alpha, \cos\alpha, 0), \qquad \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta) = (0, 0, \cos\theta),$$

$$\frac{\partial\widetilde{\phi}}{\partial\alpha} \times \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta) = \cos\theta(\cos\alpha, \sin\alpha, 0), \qquad \left\|\frac{\partial\widetilde{\phi}}{\partial\alpha} \times \frac{\partial\widetilde{\phi}}{\partial\theta}(\alpha, \theta)\right\| = \cos\theta.$$

Therefore $f(V) = \widetilde{\phi}(D)$ implies

$$\text{area}\,\big(f(V)\big) = \int_D \cos\theta \, d\alpha d\theta.$$

(iii) The assertion is a direct consequence of Exercise 3.6 on cylindrical coordinates.

(iv) If $W \subset C^2$, then $W = \psi(D)$, for some D as in part (ii), while

$$\frac{\partial\psi}{\partial\alpha}(\alpha, x_3) = (-\sin\alpha, \cos\alpha, 0), \qquad \frac{\partial\psi}{\partial x_3}(\alpha, x_3) = (0, 0, 1),$$

$$\frac{\partial\psi}{\partial\alpha} \times \frac{\partial\psi}{\partial x_3}(\alpha, x_3) = (\cos\alpha, \sin\alpha, 0), \qquad \left\|\frac{\partial\psi}{\partial\alpha} \times \frac{\partial\psi}{\partial x_3}(\alpha, x_3)\right\| = 1,$$

This and the fact that $g(W) = D$ now yield

$$\text{area}(W) = \int_D d\alpha dx_3 = \text{area}\,\big(g(W)\big).$$

(v) We may assume that the great circles intersect at the poles of S^2, since this can be achieved by applying a rotation of $\mathbf{R}^3$, which is area-preserving. Now the image $f(V)$ is a curved rectangle on C^2 of width α and height 2. Next unroll S^2 on the plane $\mathbf{R}^2$, in other words, apply g. Then the curved rectangle will be mapped to a genuine rectangle in $\mathbf{R}^2$ of width α and height 2. Application of parts (ii) and (iv) now yields that the area of V equals 2α. In particular, S^2 is the spherical diangle of angle 2π, which implies that its area is 4π.