Exercise 0.1 (Area of n-gon). Let $\Omega \subset \mathbf{R}^2$ be the bounded open subset bounded by the n-gon with successive vertices $x^{(1)}, \dots, x^{(n)} \in \mathbf{R}^2$ in counterclockwise orientation. Taking the upper indices cyclically modulo n, one has

$$(\star) \qquad \operatorname{area}(\Omega) = \frac{1}{2} \sum_{1\le k\le n} \big(x_1^{(k+1)} + x_1^{(k)}\big)\big(x_2^{(k+1)} - x_2^{(k)}\big).$$

(i) Prove $(\star)$ by means of application of Green's Integral Theorem to Ω and the vector field $f : \mathbf{R}^2 \to \mathbf{R}^2$ with $f(x) = (0, x_1)$.

(ii) Write the n-gon as a union of n triangles and deduce

$$\sum_{1\le k\le n} \big(x_1^{(k+1)} + x_1^{(k)}\big)\big(x_2^{(k+1)} - x_2^{(k)}\big) = \sum_{1\le k\le n} \big(x_1^{(k)} x_2^{(k+1)} - x_1^{(k+1)} x_2^{(k)}\big).$$

Verify this identity also by rewriting its left-hand side.

Solution of Exercise 0.1

(i) We have $\operatorname{curl} f(x) = 1$, for all $x \in \mathbf{R}^2$, hence Green's Integral Theorem 8.3.5 implies

$$\operatorname{area}(\Omega) = \int_\Omega \operatorname{curl} f(x)\, dx = \int_{\partial\Omega} \langle\, f(y), d_1 y \,\rangle = \sum_{1\le k\le n} \int_{\partial\Omega_k} \langle\, f(y), d_1 y \,\rangle,$$

$$\text{where} \qquad \partial\Omega_k = \{\, y^{(k)}(t) := x^{(k)} + t(x^{(k+1)} - x^{(k)}) \in \mathbf{R}^2 \mid 0 \le t \le 1 \,\}.$$

As a consequence,

$$f \circ y^{(k)}(t) = (0, x_1^{(k)} + t(x_1^{(k+1)} - x_1^{(k)})), \qquad D y^{(k)}(t) = x^{(k+1)} - x^{(k)},$$

$$\langle\, f \circ y^{(k)}(t), D y^{(k)}(t) \,\rangle = (x_2^{(k+1)} - x_2^{(k)})(x_1^{(k)} + t(x_1^{(k+1)} - x_1^{(k)})),$$

$$\int_{\partial\Omega_k} \langle\, f(y), d_1 y \,\rangle = (x_2^{(k+1)} - x_2^{(k)}) \int_0^1 (x_1^{(k)} + t(x_1^{(k+1)} - x_1^{(k)}))\, dt$$

$$= \big(x_2^{(k+1)} - x_2^{(k)}\big)\big(x_1^{(k)} + \frac{1}{2}(x_1^{(k+1)} - x_1^{(k)})\big) = \frac{1}{2}\big(x_1^{(k+1)} + x_1^{(k)}\big)\big(x_2^{(k+1)} - x_2^{(k)}\big).$$

(ii) Write Ω as a union of n triangles with vertices 0, $x_1^{(k)}$ and $x_1^{(k+1)}$, for $1 \le k \le n$. Next, note that the area of such a triangle equals half the area of the parallelogram spanned by the vectors $x_1^{(k)}$ and $x_1^{(k+1)}$, where the latter area is given by

$$\begin{vmatrix} x_1^{(k)} & x_1^{(k+1)} \\ x_2^{(k)} & x_2^{(k+1)} \end{vmatrix} = x_1^{(k)} x_2^{(k+1)} - x_1^{(k+1)} x_2^{(k)}.$$

For another proof of the identity in part (ii), expand the products at its left–hand side and observe that

$$\sum_{1\le k\le n} x_1^{(k+1)} x_2^{(k+1)} - \sum_{1\le k\le n} x_1^{(k)} x_2^{(k)} = 0.$$