Exercise 0.1 (Geometry of quadratic equation). In this exercise we consider the polynomial function $p(x,y) = x^2 + 2y_1x + y_2$ in the real variable x with real coefficients $2y_1$ and y_2 as a function $p : \mathbf{R}^3 \to \mathbf{R}$ of all three variables simultaneously, thus

$$p : \mathbf{R} \times \mathbf{R}^2 \simeq \mathbf{R}^3 \to \mathbf{R} \qquad \text{given by} \qquad p(x,y) = x^2 + 2y_1x + y_2.$$

Various properties of the quadratic equation can be read off from geometric properties of the zero-set

$$N = \{\, (x,y) \in \mathbf{R} \times \mathbf{R}^2 \mid p(x,y) = 0 \,\},$$

and vice versa. Figure 1 below shows the smooth surface N in $\mathbf{R}^3$; such a surface is called a *hyperbolic paraboloid*.

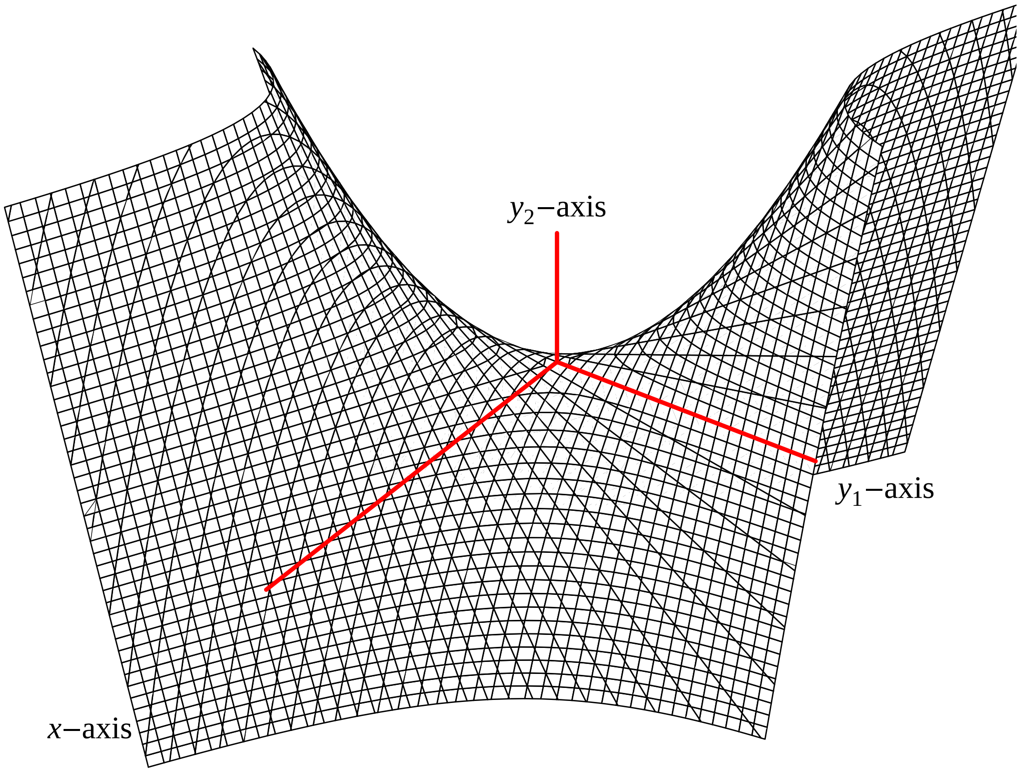

Figure 1: Hyperbolic paraboloid

In turn, the illustration immediately raises new questions: we see that N contains downward parabolae in planes perpendicular to the y_1-axis as well as hyperbolae in planes perpendicular to the y_2-axis. In this exercise we will study these more closely. We begin by surveying some of the well-known algebraic aspects.

(i) Prove

$$p(x,y) = (x + y_1)^2 - \Delta(y) \qquad \text{where} \qquad \Delta(y) = y_1^2 - y_2;$$

in fact, $\Delta : \mathbf{R}^2 \to \mathbf{R}$ is the *discriminant* of the quadratic equation. Now suppose that $(x,y) \in \mathbf{R}^3$ satisfies $p(x,y) = 0$. Deduce that

$$(\star) \qquad \Delta(y) = (x + y_1)^2 \geq 0$$

and that there exist at most two distinct solutions x to $p(x,y) = 0$. Furthermore, conclude that x is a solution of multiplicity 2 of $p(x,y) = 0$ if and only if

$$(x, y_1) \in S := \{\, (x, y_1) \in \mathbf{R}^2 \mid x + y_1 = 0 \,\}.$$

(ii) Verify that $N = \mathrm{im}(\phi)$ where

$$\phi : \mathbf{R}^2 \to \mathbf{R}^3 \qquad \text{is defined by} \qquad \phi(x, y_1) = (x, y_1, -x^2 - 2y_1 x).$$

Deduce that N is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2.

(iii) Compute the rank of $Dp(x, y) \in \mathrm{Lin}(\mathbf{R}^3, \mathbf{R})$, for all $(x, y) \in \mathbf{R}^3$. Now prove once more, but by a method different from the one employed in part (ii), that N is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2.

Denote by $\pi : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ the orthogonal projection $\pi(x, y) = y$ and define

$$\Phi = \pi \circ \phi : \mathbf{R}^2 \to \mathbf{R}^2; \qquad \text{i.e.,} \qquad \Phi(x, y_1) = \begin{pmatrix} y_1 \\ -x^2 - 2y_1 x \end{pmatrix}.$$

(iv) Compute $D\Phi(x, y_1) \in \mathrm{End}(\mathbf{R}^2)$ as well as $\det D\Phi(x, y_1)$. Show that the set of singular points of Φ is equal to the straight line S as defined in part (i). Verify that the rank of $D\Phi(x, y_1)$ is equal to 1, for all $(x, y_1) \in S$.

In Figure 2 below we see the image set of Φ. Obviously, it has been obtained by projection of the surface from Figure 1 onto the y-plane.

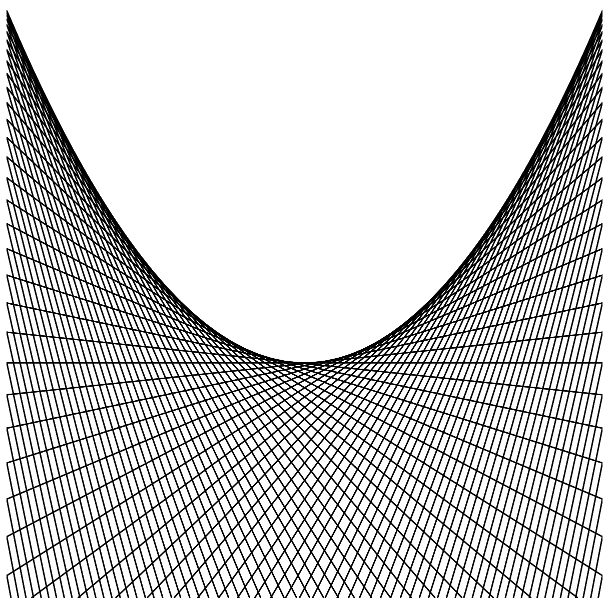

Figure 2: $\mathrm{im}(\Phi)$

(v) Prove that the image $\Phi(S) \subset \mathbf{R}^2$ equals the upward parabola (in the notation from part (i))

$$P = \{\, y \in \mathbf{R}^2 \mid \Delta(y) = 0 \,\}.$$

Furthermore, verify

$$\Phi(\mathbf{R}^2 \setminus S) = \{\, y \in \mathbf{R}^2 \mid \Delta(y) > 0 \,\};$$

in other words, this image consists of the open subset of $\mathbf{R}^2$ consisting of elements lying below the parabola P. Prove also on the basis of part (i) that $\Phi^{-1}(\{y\}) \subset \mathbf{R}^2$ always consists of two elements if $y \in \Phi(\mathbf{R}^2 \setminus S)$. Translate these results into an assertion about the intersection of N by straight lines parallel to the x-axis.

(vi) On the basis of part (v) show that $\phi(S) = \pi^{-1}(P)$ and also that this set equals the space curve

$$\Sigma = \mathrm{im}(\sigma) \subset N \qquad \text{with} \qquad \sigma : \mathbf{R} \to \mathbf{R}^3 \qquad \text{given by} \qquad \sigma(x) = (x, -x, x^2).$$

In Figure 3 below we see the plane curve Σ. Prove

$$\Sigma = \{ (x, y) \in \mathbf{R} \times \mathbf{R}^2 \mid p(x, y) = D_1 p(x, y) = 0 \}.$$

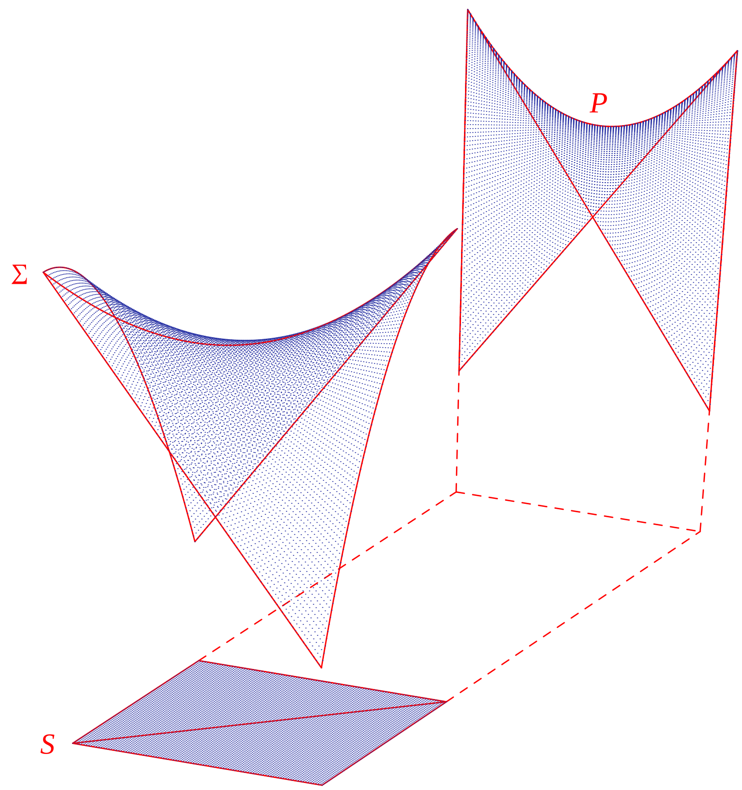

Figure 3: The straight line S, the plane curve Σ and the parabola P

(vii) Verify that the intersection of N with a plane $\{ (x, y) \in \mathbf{R} \times \mathbf{R}^2 \mid y_1 \text{ is constant} \}$ (i.e., a plane perpendicular to the y_1-as) is a downward parabola having its vertex at the point $\sigma(-y_1)$.

(viii) Give a parametrization of the geometric tangent line $\Lambda(x)$ of the curve Σ at the point $\sigma(x)$, for every $x \in \mathbf{R}$.

In the Figures 1 and 4 we also see straight lines running on the surface N in planes that appear to be perpendicular to the x-axis. We will prove the existence of such lines. To this end, let $x \in \mathbf{R}$ be fixed and define $N(x)$ to be the orthogonal projection of $\Lambda(x)$ onto the plane $\{ (x, y) \in \mathbf{R}^3 \mid y \in \mathbf{R}^2 \}$ (that is, the plane passing through $\sigma(x)$ and perpendicular to the x-axis).

(ix) Verify that $N(x)$ is the straight line $\sigma(x) + \mathbf{R}(0, -1, 2x)$ and that the surface N is the disjoint union of the lines $N(x)$, for all $x \in \mathbf{R}$. Show that every line $N(x)$ intersects the curve Σ in exactly one point.

Background. Given $x \in \mathbf{R}$, the line $N(x)$ parametrizes all quadratic equations with prescribed zero x while $\sigma(x)$ represents the unique quadratic equation having the zero x occurring with multiplicity two.

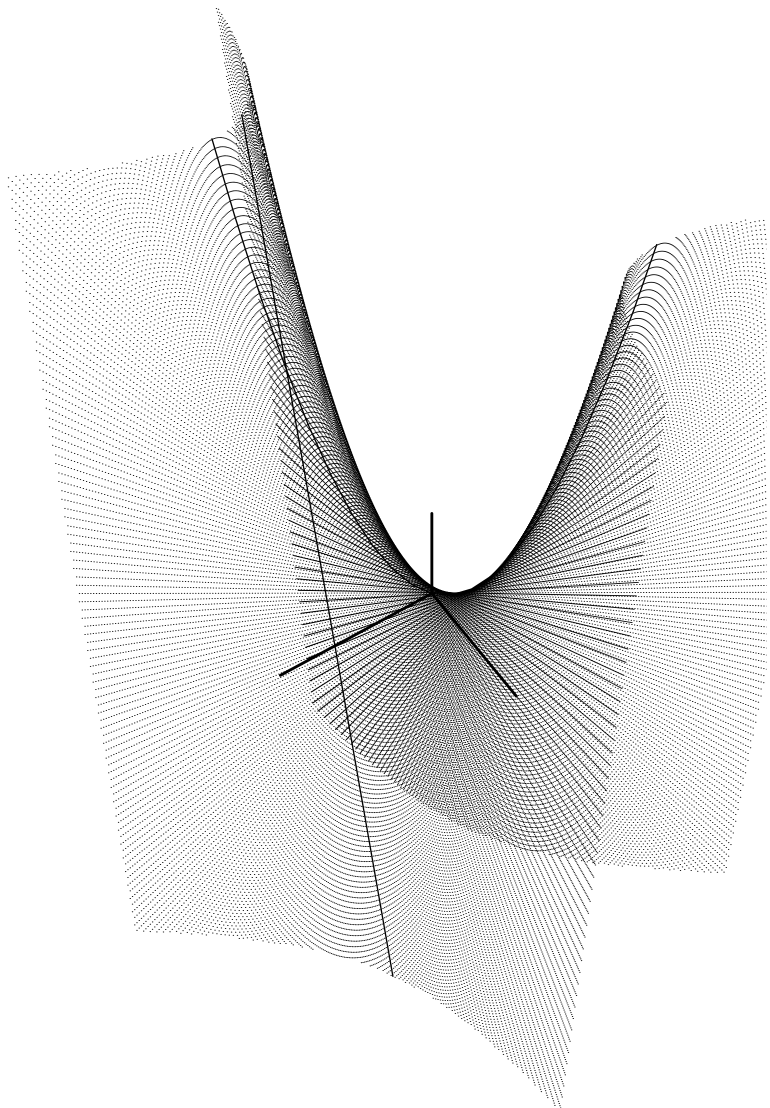

Figure 4

(x) Introduce the numbers $\varphi_\pm = \frac{1}{2}(1 \pm \sqrt{5})$ and note that φ_+ equals the *golden ratio* satisfying $-\varphi_+\varphi_- = 1$. Next define the matrices

$$O = \frac{1}{\sqrt[4]{5}} \begin{pmatrix} \sqrt{\varphi_+} & \sqrt{-\varphi_-} \\ \sqrt{-\varphi_-} & -\sqrt{\varphi_+} \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix}.$$

Show that O is both symmetric and orthogonal, while

$$O^t A O = D.$$

In addition, introduce new coordinates z in $\mathbf{R}^3$ by means of

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = O^t \begin{pmatrix} x \\ y_1 \end{pmatrix} \qquad \text{and} \qquad z_3 = y_2,$$

and deduce using Formula (2.29)

$$x^2 + 2y_1x + y_2 = \varphi_+ z_1^2 + \varphi_- z_2^2 + z_3 = \frac{1}{2}(\sqrt{5}+1)z_1^2 - \frac{1}{2}(\sqrt{5}-1)z_2^2 + z_3.$$

Recall the classification of quadrics as discussed in linear algebra and conclude that the quadric N is a hyperbolic paraboloid.

Solution of Exercise 0.1

(i) We have

$$p(x,y) = x^2 + 2y_1x + y_1^2 - (y_1^2 - y_2) = (x+y_1)^2 - \Delta(y).$$

If $p(x,y) = 0$ then the assertion of $(\star)$ is obvious as squares are nonnegative. It follows that every solution $x \in \mathbf{R}$ of $p(x,y) = 0$ is given by $x_\pm = -y_1 \pm \sqrt{\Delta(y)}$; accordingly, maximally two do exist. Obviously $x_+ = x_-$ if and only if $\Delta(y) = 0$; hence, the final assertion is a direct consequence of $(\star)$.

(ii) The equality $p(x,y) = 0$ is equivalent with $y_2 = -x^2 - 2y_1x$, which shows that $N = \mathrm{im}(\phi)$. Furthermore, $N = \mathrm{graph}(f)$ where $f : \mathbf{R}^2 \to \mathbf{R}$ with $f(x,y_1) = -x^2 - 2y_1x$ is a C^∞ function; therefore N is a C^∞ submanifold of $\mathbf{R}^3$ of dimension 2 on the basis of Definition 4.2.1.

(iii) The identity $Dp(x,y) = (*,*,1)$ shows that the rank of $Dp(x,y)$ equals 1 everywhere; in other words, $Dp(x,y)$ is surjective, for all $(x,y) \in \mathbf{R}^3$. Hence the second assertion is a direct consequence of the Submersion Theorem 4.5.2.

(iv) Differentiation immediately yields the following formulae:

$$D\Phi(x,y_1) = \begin{pmatrix} 0 & 1 \\ -2x-2y_1 & -2x \end{pmatrix} \qquad \text{and} \qquad \det D\Phi(x,y_1) = 2(x+y_1).$$

By definition, the determinant vanishes at singular points. Hence, the identification of the set of singular points with S follows directly, whereas the equation above obviously is that of a straight line. The assertion on the rank of $D\Phi(x,y_1)$, for $(x,y_1) \in S$, follows from the fact that in this case

$$D\Phi(x,y_1) = \begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}.$$

(v) Suppose $(x,y) \in \mathbf{R}^3$ satisfies $\Phi(x,y_1) = y$. Then, in particular, we have $p(x,y) = 0$ and so we obtain from $(\star)$ in part (i) that $\Delta(y) \geq 0$. Hence the inclusions $\Phi(S) \subset P$ and $\Phi(\mathbf{R}^2 \setminus S) \subset \{\, y \in \mathbf{R}^2 \mid \Delta(y) > 0 \,\}$ are obvious on the basis of $(\star)$ again. Now we prove the reverse inclusions. According to part (i) the condition $\Delta(y) = 0$ on $y \in \mathbf{R}^2$ ensures that there is a unique solution $x \in \mathbf{R}$ for $p(x,y) = 0$, i.e., $y = \Phi(x,y_1)$; furthermore, $(\star)$ then implies that $(x,y_1) \in S$. Next, suppose $y \in \mathbf{R}^2$ satisfies $\Delta(y) > 0$. From part (i) we then obtain the existence of two different solutions $x_\pm$ of the equation $p(x,y) = 0$, and this gives two distinct elements $(x_\pm, y_1) \in \mathbf{R}^2$ both belonging to $\Phi^{-1}(\{y\})$. Using $(\star)$ once more, we actually get $(x_\pm, y_1) \in \mathbf{R}^2 \setminus S$. In geometric terms, lines in $\mathbf{R}^3$ parallel to the x-axis, which means being of the form $\{\, (x,y) \in \mathbf{R}^3 \mid x \in \mathbf{R} \,\}$, intersect the surface N once, and twice, if $\Delta(y)$ is 0, and positive, respectively, and in no other case.

(vi) By definition $\Phi = \pi \circ \phi$; hence, we obtain $\pi^{-1} \circ \Phi = \phi$ (abusing the notation for the inverse image). Application of this identity to the set S gives the equality $\phi(S) = \pi^{-1}(P)$. Next, suppose

$(x, y_1) \in S$, in other words, $y_1 = -x$. Then $\phi(x, y_1) = (x, -x, y_2) \in \phi(S) = \pi^{-1}(P)$ implies $y_2 = x^2$. Accordingly

$$\phi(x, y_1) = (x, -x, x^2) = \sigma(x), \qquad \text{that is,} \qquad \phi(S) \subset \Sigma.$$

Conversely, $(x, y) \in \Sigma$ implies

$$(x, y) = \sigma(x) = (x, -x, x^2) = \phi(x, -x), \qquad \text{i.e.,} \qquad \Sigma \subset \phi(S).$$

Now the last assertion. $(x, y) \in \Sigma$ means that x is a solution of $p(X, y) = (X - x)^2 = X^2 - 2xX + x^2 = 0$, and as a consequence x is a solution of $D_1 p(X, y) = 2(X - x)$ too. Accordingly, $p(x, y) = D_1 p(x, y) = 0$. Conversely, suppose $(x, y) \in \mathbf{R}^3$ satisfies $p(x, y) = 0$ and $D_1 p(x, y) = 2(x + y_1) = 0$; hence, in particular, $y_1 = -x$. Hence $(x, y) \in \phi(S) = \Sigma$.

(vii) If y_1 is fixed and $p(x, y) = 0$, we get from $(\star)$ in part (i)

$$y_2 = y_1^2 - \Delta(y) = y_1^2 - (x + y_1)^2.$$

The right-hand side is maximal if $x + y_1 = 0$ and if this is the case it assumes the value y_1^2. Hence the vertex of the parabola has coordinates $(-y_1, y_1, y_1^2) = \sigma(-y_1)$ and it also opens downward.

(viii) In view of $D\sigma(x) = (1, -1, 2x)$, a parametric representation for $\Lambda(x)$ is given by $\sigma(x) + \mathbf{R}(1, -1, 2x)$.

(ix) $(0, -1, 2x)$ is the orthogonal projection of $D\sigma(x)$ onto the (y_1, y_2)-plane along the x-axis; hence, $N(x)$ may be described as given. By definition, the lines $N(x)$ are disjoint, for distinct $x \in \mathbf{R}$. Furthermore, consider $(x, y) \in N(x)$, that is, satisfying $y_1 = -x - \lambda$ and $y_2 = x^2 + 2\lambda x$, for some $\lambda \in \mathbf{R}$. Then $(x, y) \in N$ as follows from

$$p(x, y) = x^2 + 2y_1 x + y_2 = x^2 - 2(x + \lambda)x + x^2 + 2\lambda x = 0.$$

Accordingly, every $N(x)$ is contained in N. Conversely, suppose $x \in \mathbf{R}$ is fixed and $(x, y) \in \mathbf{R}^3$ belongs to N. Then there exists $\lambda \in \mathbf{R}$ such that $y_1 = -x - \lambda$, while $p(x, y) = 0$ now implies

$$y_2 = -x^2 - 2y_1 x = x^2 + 2\lambda x; \qquad \text{i.e.,} \qquad (x, y) \in N(x).$$

The equality $N(x) = \sigma(x) + \mathbf{R}(0, -1, 2x)$ implies that $N(x)$ intersects Σ in $\sigma(x)$, and this is the only point of intersection as the elements of Σ are uniquely determined by their first component.

(x) Straightforward computation. The quadric N is a hyperbolic paraboloid since the corresponding quadratic form has two nonzero eigenvalues of opposite sign as well as a linear term.