Exercise 0.1 (Quintic analog of Descartes' folium). Recall that $\mathbf{R}_+ = \{ x \in \mathbf{R} \mid x > 0 \}$ and define

$$\Phi : \mathbf{R}^2_+ \to \mathbf{R}^2_+ \qquad \text{by} \qquad \Phi(x) = \frac{1}{(x_1 x_2)^2}(x_1^5, x_2^5).$$

(i) Prove that Φ is a C^∞ mapping and that $\det D\Phi(x) = 5$, for all $x \in \mathbf{R}^2_+$.

(ii) Verify that Φ is a C^∞ diffeomorphism and that its inverse is given by

$$\Psi : \mathbf{R}^2_+ \to \mathbf{R}^2_+ \qquad \text{with} \qquad \Psi(y) = (y_1 y_2)^{\frac{2}{5}}(y_1^{\frac{1}{5}}, y_2^{\frac{1}{5}}).$$

Compute $\det D\Psi(y)$, for all $y \in \mathbf{R}^2_+$.

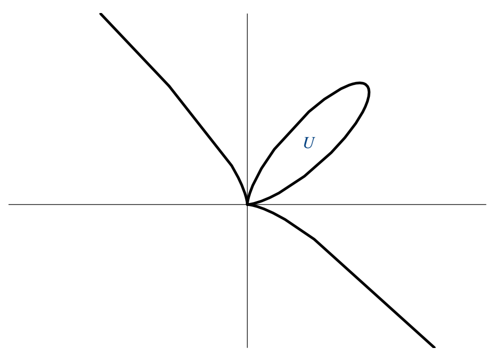
Let $a > 0$ and define

$$g : \mathbf{R}^2 \to \mathbf{R} \qquad \text{by} \qquad g(x) = x_1^5 + x_2^5 - 5a(x_1 x_2)^2.$$

Now consider the bounded open sets

$$U = \{ x \in \mathbf{R}^2_+ \mid g(x) < 0 \} \qquad \text{and} \qquad V = \{ y \in \mathbf{R}^2_+ \mid y_1 + y_2 < 5a \}.$$

Then U has a curved boundary, while V is an isosceles rectangular triangle.



(iii) Show that $g \circ \Psi(y) = (y_1 y_2)^2 (y_1 + y_2 - 5a)$, for all $y \in \mathbf{R}^2_+$. Deduce that the restriction $\Psi|_V : V \to U$ is a diffeomorphism.

Background. By means of parts (ii) and (iii) one immediately computes the area of U to be $\frac{5a^2}{2}$. Denote by F the zero-set of g (see the curve in the illustration above).

(iv) Prove that F is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 at every point of $F \setminus \{0\}$.

(v) By means of intersection with lines through 0 obtain the following parametrization of a part of F:

$$\phi : \mathbf{R} \setminus \{-1\} \to \mathbf{R}^2 \qquad \text{satisfying} \qquad \phi(t) = \frac{5at^2}{1+t^5}\begin{pmatrix} 1 \\ t \end{pmatrix}.$$

(vi) Compute that

$$\phi'(t) = \frac{5at}{(1+t^5)^2}\begin{pmatrix} 2 - 3t^5 \\ t(3 - 2t^5) \end{pmatrix}.$$

Show that ϕ is an immersion except at 0.

(vii) Demonstrate that F is not a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 at 0.

For $|x_2|$ small, x_2^5 is negligible; hence, after division by the common factor x_1^2 the equation $g(x) = 0$ takes the form $x_1^3 = 5ax_2^2$, which is the equation of an ordinary cusp. This suggests that F has a cusp at 0.

(viii) Prove that F actually possesses two cusps at 0. This can be done with simple calculations; if necessary, however, one may use without proof

$$\begin{aligned}\phi''(t) &= \frac{10a}{(1+t^5)^3}\begin{pmatrix} 6t^{10}-18t^5+1 \\ t(3t^{10}-19t^5+3)\end{pmatrix}, \\ \phi'''(t) &= -\frac{30a}{(1+t^5)^4}\begin{pmatrix} 5t^4(2t^{10}-16t^5+7) \\ 4t^5(t^{10}-17t^5+13)-1\end{pmatrix}.\end{aligned}$$

Solution of Exercise 0.1

(i) Φ is a composition of C^∞ mappings. We have

$$D\Phi(x) = \begin{pmatrix} 3\Big(\dfrac{x_1}{x_2}\Big)^2 & -2\Big(\dfrac{x_1}{x_2}\Big)^3 \\ -2\Big(\dfrac{x_2}{x_1}\Big)^3 & 3\Big(\dfrac{x_2}{x_1}\Big)^2 \end{pmatrix} \qquad \text{and so} \qquad \det D\Phi(x) = 9-4 = 5.$$

(ii) Given arbitrary $y \in \mathbf{R}^2_+$, consider the equation $\Phi(x) = y$ for $x \in \mathbf{R}^2_+$; then $\frac{x_1^3}{x_2^2} = y_1$ and $\frac{x_2^3}{x_1^2} = y_2$. Multiplication and division of these equalities leads to

$$x_1x_2 = y_1y_2 \qquad \text{and} \qquad \Big(\frac{x_1}{x_2}\Big)^5 = \frac{y_1}{y_2}. \qquad \text{So} \qquad x_1x_2 = y_1y_2 \qquad \text{and} \qquad \frac{x_1}{x_2} = \frac{y_1^{\frac15}}{y_2^{\frac15}},$$

and multiplication of the equalities now gives $x_1^2 = y_1^{\frac65}y_2^{\frac45}$. Accordingly, $x_1 = y_1^{\frac35}y_2^{\frac25} = (y_1y_2)^{\frac25}y_1^{\frac15}$ because x_1, y_1 and $y_2 \in \mathbf{R}_+$. Similarly, we obtain the desired formula for x_2. It follows that Φ and Ψ are each other's inverses. On $\mathbf{R}^2_+$ the mapping Ψ is of class C^∞, which implies that Φ is a C^∞ diffeomorphism. From part (i) and the multiplicative property of the determinant we obtain $\det D\Psi(y) = \frac15$.

(iii) We find

$$g\circ\Psi(y) = (y_1y_2)^2(y_1+y_2) - 5a(y_1y_2)^{\frac85}(y_1y_2)^{\frac25} = (y_1y_2)^2(y_1+y_2-5a).$$

This implies $x = \Psi(y) \in U$ if and only if $g(x) = (y_1y_2)^2(y_1+y_2-5a) < 0$ if and only $y_1+y_2-5a < 0$ if and only if $y \in V$.

(iv) We have

$$Dg(x) = 5(x_1(x_1^3-2ax_2^2),\ x_2(x_2^3-2ax_1^2)).$$

This matrix is of rank 1 unless (a) $x = 0$ or (b) $x_1^3 = 2ax_2^2$ and $x_2^3 = 2ax_1^2$. In case (b) we may assume $x \neq 0$ and we also obtain $x_1^9 = 8a^3x_2^6 = 32a^5x_1^4$, that is, $x_1^5 = (2a)^5$, which holds if and only if $x_1 = 2a$. In turn this implies $x_2 = 2a$, but $g(2a,2a) = 64a^5 - 80a^5 = -16a^5 < 0$; in other words, $(2a,2a) \notin F$. It follows that g is submersive at every point of $F\setminus\{0\}$. The desired conclusion follows from the Submersion Theorem 4.5.2.(ii).

(v) We eliminate x_2 from the equations $g(x) = 0$ and $x_2 = tx_1$, for fixed $t \in \mathbf{R}$. This leads to $(1+t^5)x_1^5 = 5at^2x_1^4$, with solutions $x_1 = 0$ (as was to be expected) or $x_1 = \frac{5at^2}{1+t^5}$; thus the desired formula for ϕ holds.

(vi) The formula for ϕ' is a consequence of

$$\phi'(t) = \frac{5a}{(1+t^5)^2}\begin{pmatrix} 2t\,(1+t^5) - t^2\,5t^4 \\ 3t^2(1+t^5) - t^3\,5t^4 \end{pmatrix} = \frac{5at}{(1+t^5)^2}\begin{pmatrix} 2+2t^5-5t^5 \\ t(3+3t^5-5t^5) \end{pmatrix}.$$

If $t \neq 0$, then the assumption $\phi'(t) = 0$ implies $2 - 3t^5 = 0$ and $3 - 2t^5 = 0$. This gives $9t^5 = 6 = 4t^5$, that is $5t^5 = 0$, and so arrived at a contradiction. Therefore $\phi'(t) \neq 0$ if $t \neq 0$; hence $\phi'(t)$ is of rank 1, which proves that ϕ is everywhere immersive except at 0.

(vii) F has self-intersection at 0 as follows from $\lim_{t\to\pm\infty} \phi(t) = 0 = \phi(0)$. Indeed, $\widetilde{\phi} : \mathbf{R} \setminus \{-1\} \to \mathbf{R}^2$ with $\widetilde{\phi}(u) = \phi(\frac{1}{u})$ also defines a parametrization of F. Now $\phi(t)$ approaches 0 in a vertical direction as $t \downarrow 0$, while $\widetilde{\phi}(u)$ approaches 0 in a horizontal direction as $u \downarrow 0$.

(viii) Select $t_0 > 0$ sufficiently small, that is, suppose $2 - 3t_0^5 > 0$ and $3 - 2t_0^5 > 0$. For t running from -1 to t_0, the sign of the first component $t(2-3t^5)$ of $\phi'(t)$ changes from negative to positive at $t = 0$, whereas the sign of the second component $t^2(3-2t^5)$ remains nonnegative and vanishes for $t = 0$ only. This behavior of ϕ' near 0 is characteristic for a vertical cusp of F at 0. Mutatis mutandis, the same argument applied to $\widetilde{\phi}$ gives the existence of a second, horizontal, cusp of F at 0. Alternatively, it follows that

$$\phi''(0) = 10a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \phi'''(0) = 30a\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

According to Definition 5.3.9 this implies the existence of an ordinary cusp of F at 0.