Exercise 0.1 (Two-step recurrences for hyperarea and volume). Write S^{n-1} and B^n for the unit sphere and the interior of the unit ball in $\mathbf{R}^n$, respectively, and set

$$a_{n-1} = \mathrm{hyperarea}_{n-1}(S^{n-1}) \qquad \text{and} \qquad v_n = \mathrm{vol}_n(B^n).$$

Here is a table of these numbers for low values of n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_{n-1}	2	2π	4π	$2\pi^2$	$\frac{8\pi^2}{3}$	π^3	$\frac{16\pi^3}{15}$	$\frac{\pi^4}{3}$	$\frac{32\pi^4}{105}$	$\frac{\pi^5}{12}$	$\frac{64\pi^5}{945}$	$\frac{\pi^6}{60}$	$\frac{128\pi^6}{10395}$	$\frac{\pi^7}{360}$
v_n	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$	$\frac{8\pi^2}{15}$	$\frac{\pi^3}{6}$	$\frac{16\pi^3}{105}$	$\frac{\pi^4}{24}$	$\frac{32\pi^4}{945}$	$\frac{\pi^5}{120}$	$\frac{64\pi^5}{10395}$	$\frac{\pi^6}{720}$	$\frac{128\pi^6}{135135}$	$\frac{\pi^7}{5040}$

(i) In the table we see $a_{n-1} = n\, v_n$, for $1 \le n \le 14$. Prove this identity for all $n \in \mathbf{N}$, for instance, by applying Gauss' Divergence Theorem.

The table also suggests that the powers of π are given by the integral part of half the dimension and, furthermore, that there exist two-step recurrences

$$(\star) \qquad a_{n-1} = \frac{2\pi}{n-2}\, a_{n-3} \qquad \text{and} \qquad v_n = \frac{2\pi}{n}\, v_{n-2}.$$

In the following we will prove these identities geometrically (that is, without analyzing values of the Gamma function), for all $n \in \mathbf{N}$ sufficiently large. To this end, define the function $s : B^{n-2} \to \mathbf{R}_+$ by $s(x) = \sqrt{1 - \|x\|^2}$ and the mapping

$$\phi : D := B^{n-2} \times \,]-\pi, \pi[\, \to \mathbf{R}^n \qquad \text{by} \qquad \phi(x, \alpha) = \begin{pmatrix} x \\ s(x)\cos\alpha \\ s(x)\sin\alpha \end{pmatrix}.$$

(ii) Firstly, consider the case of $n = 3$. Prove that ϕ is injective and that $\mathrm{im}(\phi) = S^2$ except for a set which is negligible for 2-dimensional integration. Note that ϕ induces the mapping

$$\psi : C^2 := B^1 \times S^1 \to S^2 \qquad \text{given by} \qquad \psi(x, y) = \phi(x, \arg(y)) = \begin{pmatrix} x \\ s(x) y_1 \\ s(x) y_2 \end{pmatrix}.$$

Show that ψ is a bijection between the cylinder C^2 and the sphere minus two points. Furthermore, describe ψ in geometric terms, that is, as a projection (the inverse of ψ is known as *Lambert's cylindrical projection* of the sphere onto a tangent cylinder, see the next page for an illustration).

(iii) Next, consider the case of general $n \ge 3$. Prove $D_j s(x) = -\frac{x_j}{s(x)}$, for $1 \le j \le n-2$ and $x \in B^{n-2}$. Furthermore, write I_{n-2} for the identity matrix in $\mathrm{Mat}(n-2, \mathbf{R})$ and also x^t for the row vector obtained from $x \in B^{n-2}$ by means of transposition. Show that, for all $(x, \alpha) \in D$,

$$D\phi(x, \alpha) \in \mathrm{Lin}(\mathbf{R}^{n-1}, \mathbf{R}^n) \qquad \text{and} \qquad D\phi(x, \alpha)^t\, D\phi(x, \alpha) \in \mathrm{End}(\mathbf{R}^{n-1})$$

has the following matrix, respectively:

$$\begin{pmatrix} I_{n-2} & 0_{n-2} \\ -\dfrac{\cos\alpha}{s(x)} x^t & -s(x)\sin\alpha \\ -\dfrac{\sin\alpha}{s(x)} x^t & s(x)\cos\alpha \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} I_{n-2} + \dfrac{1}{s(x)^2} x x^t & 0 \\ 0^t & s(x)^2 \end{pmatrix}.$$

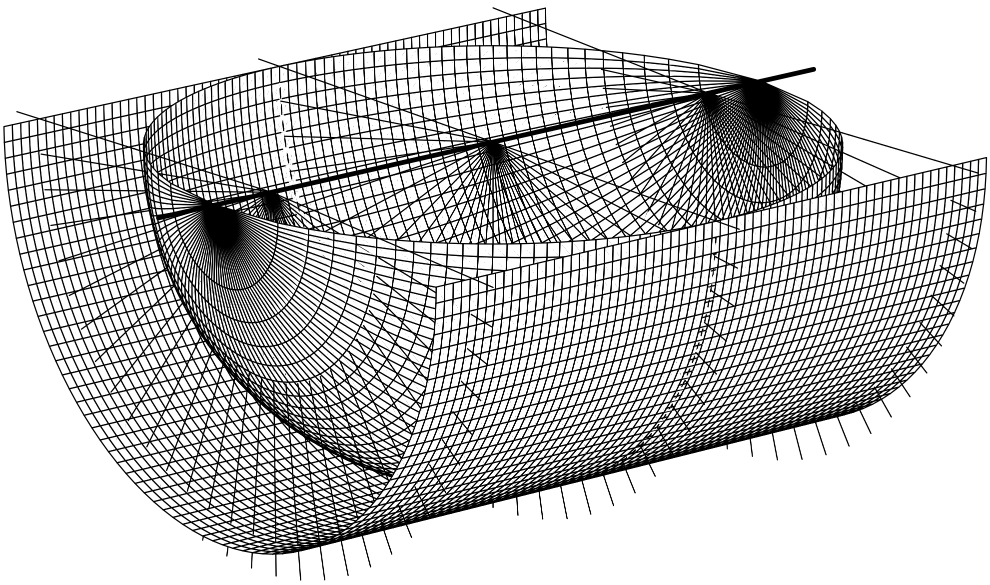

Illustration for part (ii): Lambert's projection from sphere onto tangent cylinder

(iv) Generalize the results from part (ii). Specifically, applying results from part (iii), verify that ϕ is a C^∞ embedding having an open part of S^{n-1} with negligible complement as an image.

(v) By considering the behavior of the following determinant (see part (iii)) under rotations of the element $x \in B^{n-2}$, show

$$\det\Big(I_{n-2} + \frac{1}{s(x)^2}xx^t\Big) = \frac{1}{s(x)^2} \qquad \text{and deduce} \qquad \omega_\phi(x,\alpha) = 1,$$

where ω_ϕ is the Euclidean density function associated with $\phi : D \to S^{n-1}$.

(vi) On the basis of parts (v) and (i) prove the first equality in $(\star)$ and then deduce the second one. In particular, prove by mathematical induction over $n \in \mathbf{N}$

$$v_{2n} = \frac{\pi^n}{n!}, \qquad v_{2n-1} = \frac{2^{2n}\,\pi^{n-1}\,n!}{(2n)!} \qquad \text{and} \qquad a_{2n-1} = \frac{2\pi^n}{(n-1)!}.$$

Next, we use the formula for v_{2n} in order to compute the volume of the standard $(n+1)$-tope Δ^n in $\mathbf{R}^n$ given by

$$\Delta^n = \{\, y \in \mathbf{R}^n_+ \mid \sum_{1\le j\le n} y_j < 1 \,\}. \qquad \text{In fact, we claim} \qquad (\star\star) \qquad \mathrm{vol}_n(\Delta^n) = \frac{1}{n!}.$$

For proving this, introduce

$$\Psi : \Delta^n \times \,]-\pi,\pi\,[\,^n \to B^{2n} \qquad \text{with} \qquad \Psi(y,\alpha) = \begin{pmatrix} \sqrt{y_1}\cos\alpha_1 \\ \sqrt{y_1}\sin\alpha_1 \\ \vdots \\ \sqrt{y_n}\cos\alpha_n \\ \sqrt{y_n}\sin\alpha_n \end{pmatrix}.$$

(vii) Show that Ψ is a C^∞ diffeomorphism onto an open dense subset of B^{2n} with Jacobi determinant in absolute value equal to 2^{-n} and deduce $(\star\star)$.

Background. The preceding results imply that B^{2n} is diffeomorphic with the Cartesian product of n circles with a polytope of dimension n. Analogously, B^{2n+1} is diffeomorphic with the Cartesian product of n circles with the segment of the circular paraboloid of dimension $n+1$ given by

$$\{\, (y, z) \in \mathbf{R}^n_+ \times \mathbf{R} \mid \sum_{1\le j\le n} y_j + z^2 < 1 \,\}$$

In v_n there occur as many factors π as there are independent ways to turn around in space, that is, the number of linearly independent (two-dimensional) planes. Phrased differently, the powers of π are given by the integral part of half the dimension.

(viii) According to the table above or the illustration below the sequence $(a_n)_{n=0}^{6}$ is strictly monotonically increasing while $a_6 > a_7 > a_8$. Combine these facts with $(\star)$ to prove that $(a_n)_{n=6}^{\infty}$ is strictly monotonically decreasing. Then apply part (vi) to show that $\lim_{n\to\infty} a_n = 0$. Deduce that also $(v_n)_{n=5}^{\infty}$ is strictly monotonically decreasing with $\lim_{n\to\infty} v_n = 0$.
Hint: One might use the following consequence of $(\star)$:

$$a_{n-1} = \frac{2\pi}{n-2}\frac{2\pi}{n-4}\cdots \begin{cases} \dfrac{2\pi}{7}a_6, & n \ge 7 \text{ odd;} \\[2mm] \dfrac{2\pi}{8}a_7, & n \ge 8 \text{ even.} \end{cases}$$

Accordingly, $a_6 = 33.073\cdots$ is the absolute maximum over all dimensions of the hyperareas of the corresponding unit spheres while $v_5 = 5.263\cdots$ is the absolute maximum over all dimensions of the volumes of the corresponding unit balls.

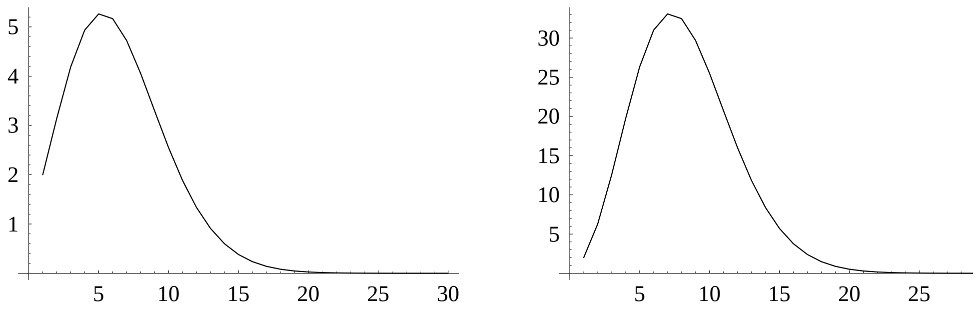

Illustration: Hyperarea a_{n-1} of unit sphere and volume v_n of unit ball, for $1 \le n \le 30$

Solution of Exercise 0.1

(i) See Example 7.9.1.

(ii) $\phi(x, \alpha) = \phi(x', \alpha')$ implies by projection onto the first coordinate that $x = x'$. Consideration of the last two coordinates then leads to $\cos\alpha = \cos\alpha'$ and $\sin\alpha = \sin\alpha'$, that is $\alpha = \alpha'$. It is straightforward that $\mathrm{im}(\phi)$ is all of S^2 except the half-circle $\{\, (x, -s(x), 0) \in S^2 \mid |x| \le 1 \,\}$ connecting the opposite points $x_\pm := (\pm 1, 0, 0)$. The half-circle is compact and of dimension 1 which implies that it is negligible for 2-dimensional integration (see page 526). We have

$$C^2 = \{\, x \in \mathbf{R}^3 \mid |x_1| < 1,\ x_2^2 + x_3^2 = 1 \,\},$$

which shows that it is a cylinder, parallel to the x_1-axis. The preceding argument implies that ψ induces a bijection between C^2 and $S^2 \setminus \{x_\pm\}$. Given $(x, y) \in C^2$, its image $\psi(x, y) \in S^2$ may be obtained in the following geometrical manner. Denote by ℓ the unique straight line in $\mathbf{R}^3$ containing (x, y) that is parallel to the plane $\{\, x \in \mathbf{R}^3 \mid x_1 = 0 \,\}$ and that intersects the x_1-axis. Next define $\psi(x, y)$ to be the point of intersection of ℓ with S^2 of shortest distance to (x, y).

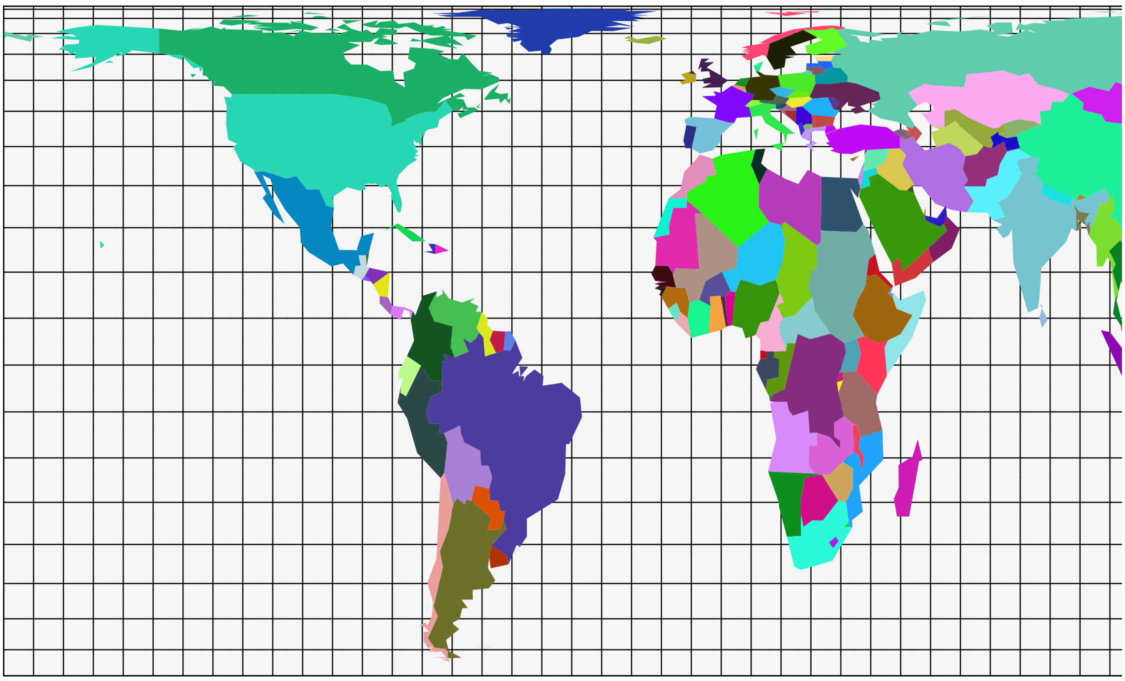

Illustration: Map of the surface of the Earth based on Lambert's cylindrical projection

(iii) On the basis of the chain rule one sees

$$D_j s(x) = \frac{1}{2s(x)}(-2x_j) = -\frac{x_j}{s(x)}; \qquad \text{in other words} \qquad \operatorname{grad} s(x) = -\frac{1}{s(x)} x^t,$$

which leads to the matrix for $D\phi(x, \alpha)$. Obviously $D\phi(x, \alpha)^t\, D\phi(x, \alpha)$ has the following matrix:

$$\begin{pmatrix} I_{n-2} & -\dfrac{\cos\alpha}{s(x)} x & -\dfrac{\sin\alpha}{s(x)} x \\ 0_{n-2} & -s(x)\sin\alpha & s(x)\cos\alpha \end{pmatrix} \begin{pmatrix} I_{n-2} & 0_{n-2} \\ -\dfrac{\cos\alpha}{s(x)} x^t & -s(x)\sin\alpha \\ -\dfrac{\sin\alpha}{s(x)} x^t & s(x)\cos\alpha \end{pmatrix}.$$

A-priori one knows the resulting matrix to be symmetric. Therefore, when multiplying the i-th row in the first matrix with the j-th column in the second, one has to distinguish only three cases: $1 \le i, j \le n-2$, which leads to the upper-left matrix belonging to $\mathrm{Mat}(n-2, \mathbf{R})$ in the answer; $i = j = n-1$, which gives the lower-right entry as a consequence of $\sin^2 + \cos^2 = 1$; and $i = n-1$ and $1 \le j \le n-2$, which leads to $\sin\alpha\cos\alpha\, x_j - \cos\alpha\sin\alpha\, x_j = 0$.

(iv) ϕ is of class C^∞ since all of its component functions are. Next $\mathrm{im}(\phi) \subset S^{n-1}$; indeed, for $(x, \alpha) \in D$,

$$\|\phi(x, \alpha)\|^2 = \|x\|^2 + s(x)^2(\cos^2\alpha + \sin^2\alpha) = \|x\|^2 + 1 - \|x\|^2 = 1.$$

Actually, $\mathrm{im}(\phi)$ is all of S^{n-1} except the set $\{ (x, -s(x), 0) \in S^{n-1} \mid x \in \overline{B^{n-2}} \}$. This set is compact and of dimension $= \dim(B^{n-2}) = n-2$; that implies that it is negligible for $(n-1)$-dimensional integration (see page 526). Furthermore, ϕ is an embedding if it is immersive, injective and has a continuous inverse upon restriction to its image. Now, suppose $h \in \mathbf{R}^{n-1}$ satisfies $\mathbf{R}^n \ni D\phi(x,\alpha)h = 0$. In view of part (iii) the upper $n-2$ entries of the image vector give $h_1 = \cdots = h_{n-2} = 0$, while the two bottom entries lead to $(\sin^2\alpha + \cos^2\alpha)h_{n-1} = h_{n-1} = 0$. Accordingly, $D\phi(x,\alpha)$ is injective, for all $(x,\alpha) \in D$. As in part (ii) one shows directly that ϕ is injective on D. Finally, if $\phi(x,\alpha) = y \in \mathbf{R}^n$, then projection of y onto its upper $n-2$ entries produces x, while $\alpha = 2\arctan\left(\frac{y_n}{1+y_{n-1}}\right)$. This implies that the inverse mapping $\phi^{-1} : \phi(D) \to D$ with $\phi(x,\alpha) \mapsto (x,\alpha)$ is continuous.

(v) Exactly the same arguments as in the solution to Exercise 6.23.(iii) imply

$$\det\Big(I_{n-2} + \frac{1}{s(x)^2}xx^t\Big) = 1 + \frac{\|x\|^2}{s(x)^2} = \frac{1}{s(x)^2}.$$

As a consequence

$$\omega_\phi(x,\alpha) = \sqrt{\det\left(D\phi(x,\alpha)^t\, D\phi(x,\alpha)\right)} = \frac{1}{s(x)}s(x) = 1.$$

(vi) $\mathrm{im}(\phi) = S^{n-1}$ up to a negligible set according to part (iv), therefore one obtains from parts (v) and (i)

$$a_{n-1} = \int_{S^{n-1}} d_{n-1}y = \int_D \omega_\phi(y)\,dy = \int_{B^{n-2}} dx \int_{-\pi}^{\pi} d\alpha = 2\pi v_{n-2} = 2\pi\frac{a_{n-3}}{n-2}.$$

This implies directly

$$v_n = \frac{1}{n}a_{n-1} = \frac{2\pi}{n}\frac{a_{n-3}}{n-2} = \frac{2\pi}{n}v_{n-2}.$$

The formulae for v_n are a direct consequence of the identities $v_2 = \pi$ and $v_1 = 2$, while the formula for a_{2n-1} follows from part (i).

(vii) It is straightforward that Ψ is a C^∞ diffeomorphism onto its image. This image consists of B^{2n} under omission of the union of the origin and of all the sets (this union is negligible for $2n$-dimensional integration)

$$\{ (x_1, \dots, x_{2j-1}, -z_j, 0, x_{2j+1}, \dots, x_{2n}) \in B^{2n} \mid 0 < z_j < 1 \} \qquad (1 \le j \le n).$$

Write $\Psi(y,\alpha) = \Psi'(y_1,\alpha_1,\cdots,y_n,\alpha_n)$. Since the difference between Ψ and Ψ' is a permutation of the coordinates, one has

$$|\det D\Psi(y,\alpha)| = |\det D\Psi'(y_1,\alpha_1,\cdots,y_n,\alpha_n)| = \prod_{1\le j\le n}\begin{vmatrix} \dfrac{\cos\alpha_j}{2\sqrt{y_j}} & -\sqrt{y_j}\sin\alpha_j \\ \dfrac{\sin\alpha_j}{2\sqrt{y_j}} & \sqrt{y_j}\cos\alpha_j \end{vmatrix} = \frac{1}{2^n}.$$

On the basis of the Change of Variables Theorem 6.6.1 it is obvious now that

$$\frac{\pi^n}{n!} = v_{2n} = \int_{B_{2n}} dx = \int_{\Delta^n \times\,]-\pi,\pi[^n} \frac{1}{2^n}\,dy\,d\alpha = \pi^n\,\mathrm{vol}_n(\Delta^n).$$

(viii) According to $(\star)$ we have

$$a_{n-1} = \frac{2\pi}{n-2}\frac{2\pi}{n-4}\cdots\begin{cases}\dfrac{2\pi}{7}a_6, & n \geq 7 \text{ odd;}\\[2ex] \dfrac{2\pi}{8}a_7, & n \geq 8 \text{ even.}\end{cases}$$

Now, for $n \geq 4$,

$$\frac{2\pi}{2n-2}\frac{2\pi}{2n-4}\cdots\frac{2\pi}{7}a_6 > \frac{2\pi}{2n-1}\frac{2\pi}{2n-3}\cdots\frac{2\pi}{8}a_7 > \frac{2\pi}{2n}\frac{2\pi}{2n-2}\cdots\frac{2\pi}{9}a_8,$$

which together with the preceding assertion leads to the desired strict monotonicity

$$a_{2n-1} > a_{2n} > a_{2n+1}.$$

According to part (vi), for $n \geq 4$,

$$0 < a_{2n-1} = \frac{2\pi^n}{(n-1)!} = 2\pi \prod_{1\leq k<n} \frac{\pi}{k} \leq \frac{\pi^4}{3} \prod_{4\leq k<n} \frac{\pi}{4} = \frac{\pi^4}{3}\left(\frac{\pi}{4}\right)^{n-4}.$$

As $\frac{\pi}{4} < 1$, this implies $\lim_{n\to\infty} a_{2n-1} = 0$, which gives $\lim_{n\to\infty} a_n = 0$ in view of the preceding result. Applying part (i) we get the desired monotonicity for $(v_n)_{n=7}^{\infty}$; and, as a consequence, for $(v_n)_{n=5}^{\infty}$ too because $v_5 > v_6 > v_7$ can be gleaned from the table. Furthermore, the limit statement for the v_n follows directly from the one for the a_n, again on the basis of part (i).