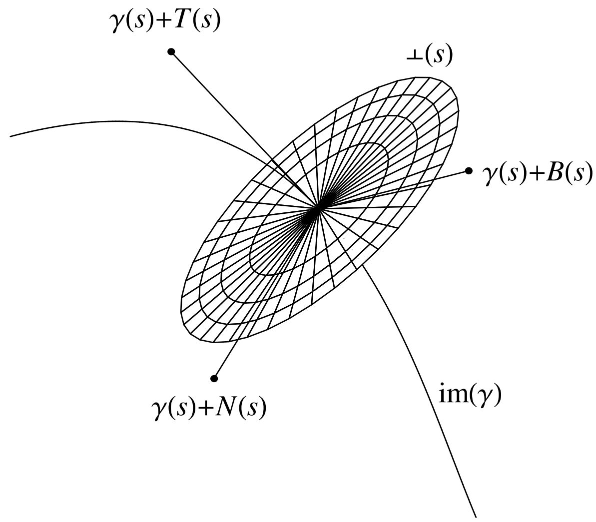
Exercise 0.1 (Formulae of Serret–Frenet and tubular neighborhood of curve). Let $J \subset \mathbf{R}$ be an open interval in $\mathbf{R}$ and let $\gamma : J \to \mathbf{R}^3$ be a C^∞ curve in $\mathbf{R}^3$. For any $s \in J$, denote by $\perp(s)$ the plane in $\mathbf{R}^3$ that contains the point $\gamma(s)$ and is perpendicular to the tangent vector $T(s) := \gamma'(s) \in \mathbf{R}^3$ of $\mathrm{im}(\gamma)$ at $\gamma(s)$. In this exercise, $'$ denotes the derivative of a mapping defined on J with respect to the variable in J.



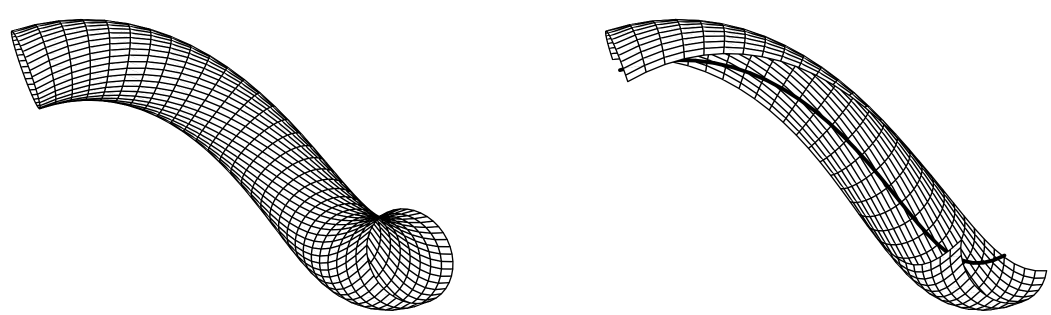
(i) Prove $\perp(s) = \{\, x \in \mathbf{R}^3 \mid \langle\, x - \gamma(s),\, T(s)\,\rangle = 0 \,\}$.

(ii) Consider $x \in \mathbf{R}^3$ and suppose the function $s \mapsto \|x - \gamma(s)\|$ attains a minimum at $s_0 \in J$. Show $x \in \perp(s_0)$.

Now suppose that γ be parametrized by arc length, in other words, that $\|T(s)\| = 1$, and furthermore, that $\gamma''(s) \neq 0$, for all $s \in J$. Consider the mutually perpendicular unit vectors $T(s)$, $N(s)$ and $B(s) \in \mathbf{R}^3$ from Definition 5.8.1.

(iii) Deduce that $N(s) \times B(s) = T(s)$ and $B(s) \times T(s) = N(s)$, for all $s \in J$.

(iv) Show that $\perp(s) = \{\, \gamma(s) + \lambda_1 N(s) + \lambda_2 B(s) \in \mathbf{R}^3 \mid \lambda \in \mathbf{R}^2 \,\}$.



Tubular surface.

Define $\mathrm{tub}(r)$, the *tubular surface* at a distance $r > 0$ from the curve γ, by means of

$$\mathrm{tub}(r) := \bigcup_{s \in J} \mathrm{tub}(s, r) := \bigcup_{s \in J} \{\, x \in \perp(s) \mid \|x - \gamma(s)\| = r \,\}.$$

(v) Prove that $\mathrm{tub}(r) = \mathrm{im}(\phi)$ where

$$\phi : J \times \,]-\pi,\, \pi] \to \mathbf{R}^3 \qquad \text{is given by} \qquad \phi(s, \alpha) = \gamma(s) + r\cos\alpha\, N(s) + r\sin\alpha\, B(s).$$

(vi) Using the formulae of Frenet–Serret from Section 5.8 show

$$\begin{aligned}
\frac{\partial \phi}{\partial s}(s, \alpha) &= (1 - r\kappa(s)\cos\alpha)T(s) - r\tau(s)\sin\alpha\, N(s) + r\tau(s)\cos\alpha\, B(s),\\
\frac{\partial \phi}{\partial \alpha}(s, \alpha) &= -r\sin\alpha\, N(s) + r\cos\alpha\, B(s), \qquad \Big\| \frac{\partial \phi}{\partial s} \times \frac{\partial \phi}{\partial \alpha}(s, \alpha) \Big\| = r(1 - r\kappa(s)\cos\alpha).
\end{aligned}$$

(vii) Verify that ϕ is an immersion under the assumption that $\kappa(s) < \frac{1}{r}$, for all $s \in J$. Deduce that for every point in $J \times \,]-\pi,\, \pi]$ there exists a neighborhood D such that $\phi(D) \subset \mathrm{tub}(r)$ is a C^∞ submanifold in $\mathbf{R}^3$ of dimension 2.

(viii) Suppose that γ is an embedding and that, for every $x \in \mathrm{tub}(r)$, there exists a unique $s \in J$ such that $\|x - \gamma(s)\| \leq r$. Use part (ii) to prove that ϕ is an embedding.

From now on assume that γ and ϕ are embeddings and that γ is of finite length.

(ix) Conclude $\mathrm{area}_2(\mathrm{tub}(r)) = 2\pi r \,\mathrm{length}(\gamma)$.

Next, define $\mathrm{Tub}(r)$, the open *tubular neighborhood* of radius r of the curve γ, by means of

$$\mathrm{Tub}(r) := \bigcup_{0 \leq \rho < r} \mathrm{tub}(\rho).$$

(x) Prove $\mathrm{vol}_3(\mathrm{Tub}(r)) = \pi r^2 \,\mathrm{length}(\gamma)$.

Furthermore, consider the C^∞ mapping

$$\Psi : J \times \mathbf{R}^2 \to \mathbf{R}^3 \qquad \text{given by} \qquad \Psi(s, t) = \gamma(s) + t_1\, N(s) + t_2\, B(s).$$

(xi) Compute

$$\det D\Psi(s, t) = \Big\langle \frac{\partial \Psi}{\partial s}, \frac{\partial \Psi}{\partial t_1} \times \frac{\partial \Psi}{\partial t_2} \Big\rangle(s, t) = 1 - \kappa(s)\, t_1.$$

Suppose $D(s) \subset \mathbf{R}^2$ is an open and Jordan measurable set and introduce the planar sets $U(s) \subset \perp(s)$, for $s \in J$, and the solid $U \subset \mathbf{R}^3$ by

$$U(s) = \{\, \Psi(s, t) \in \mathbf{R}^3 \mid t \in D(s) \,\} \qquad \text{and} \qquad U = \bigcup_{s \in J} U(s).$$

(xii) Assume that $\Psi : \bigcup_{s\in J}\{s\} \times D(s) \to U$ is a C^∞ diffeomorphism with positive Jacobi determinant and write $a(s) = \mathrm{area}_2(U(s))$. Prove

$$\begin{aligned}
\mathrm{vol}_3(U) &= \int_J \Big(\mathrm{area}\,(D(s)) - \kappa(s) \int_{D(s)} t_1\, dt \Big)\, ds\\
&= \int_{\mathrm{im}(\gamma)} \Big(a(s) - \kappa(s) \int_{U(s)} \langle\, y - \gamma(s),\, N(s)\,\rangle\, d_2 y \Big)\, d_1 s.
\end{aligned}$$

(xiii) Apply the formula from the previous part in the case of the helix $\gamma : J =: \,]-\pi,\, \pi[\, \to \mathbf{R}^3$ as in Example 5.8.2 with $a = b = \frac{1}{2}\sqrt{2}$ and $D(s) = \,]0,\, 1[^2$, for all $s \in J$, to show that $\mathrm{vol}_3(U) = 2\pi(1 - \frac{\sqrt{2}}{4}) = 4.061\,743 \cdots$ in this case.

Background. The result in part (x) above is a very special case of a result of H. Weyl: On the volume of tubes, Amer. J. Math. **61** (1939) 461-472. This paper has been very influential in modern differential geometry. Remarkable is that the formulae in parts (ix) and (x) are independent of the amount of "twisting" of the curve $\mathrm{im}(\gamma)$.

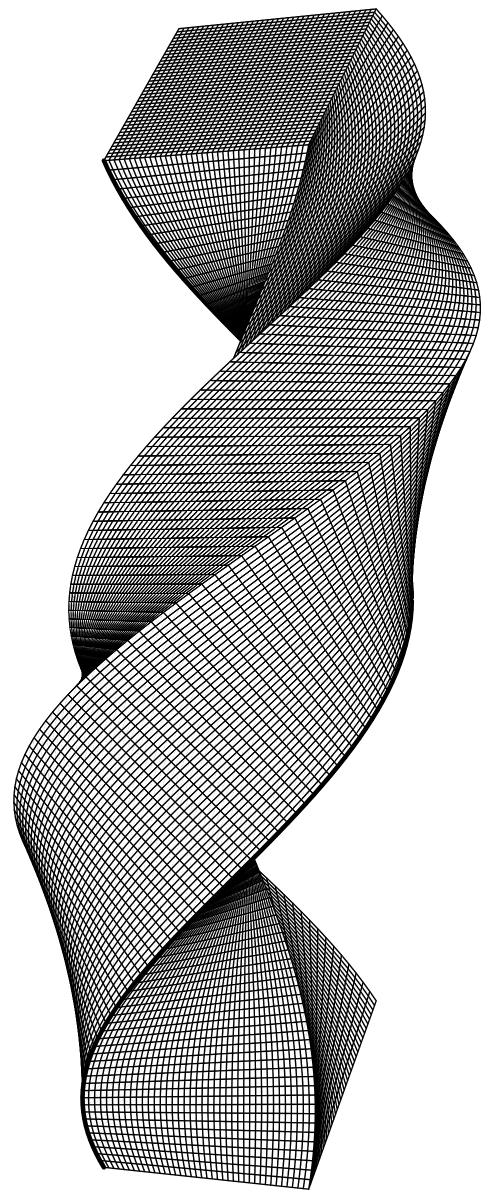


Illustration for part (ix).

Solution of Exercise 0.1

(i) Straightforward application of linear algebra.

(ii) Consider $s \mapsto \|x-\gamma(s)\|^2 = \langle x-\gamma(s),\, x-\gamma(s)\rangle$. As it attains a minimum at s_0, its derivative has to vanish at s_0, in other words, on the basis of Corollary 2.4.3.(ii)

$$\langle\, x-\gamma(s_0),\, \gamma'(s_0)\,\rangle = \langle\, x-\gamma(s_0),\, T(s_0)\,\rangle = 0, \qquad \text{that is} \qquad x \in \perp(s_0).$$

(iii) The matrix $O(s)$ from Definition 5.8.1 maps the standard basis vectors e_1, e_2 and e_3 in $\mathbf{R}^3$ to $T(s)$, $N(s)$ and $B(s)$, respectively, and being an element of $\mathbf{SO}(3,\mathbf{R})$ preserves outer products. As $e_j \times e_{j+1} = e_{j+2}$ where the indices are taken modulo 3, the desired identities follow.

(iv) $N(s)$ and $B(s)$ span the linear subspace of vectors in $\mathbf{R}^3$ perpendicular to $T(s)$.

(v) If $x = \phi(s,\alpha)$, then $x = \gamma(s) + \lambda_1 N(s) + \lambda_2 B(s) \in \perp(s)$ according to part (iv). Furthermore

$$\|x-\gamma(s)\| = r\|\cos\alpha\, N(s) + \sin\alpha\, B(s)\| = r,$$

since $N(s)$ and $B(s)$ are mutually perpendicular unit vectors. Thus, $\operatorname{im}\phi \subset \operatorname{tub}(r)$. Conversely, suppose $x \in \operatorname{tub}(r)$, then $x \in \operatorname{tub}(s,r)$, for some $s \in J$. Hence $x \in \perp(s)$ and $\|x-\gamma(s)\| = r$, that is

$$x = \gamma(s) + r\cos\alpha\, N(s) + r\sin\alpha\, B(s) = \phi(s,\alpha),$$

for some $\alpha \in\ ]-\pi,\, \pi\,]$. Therefore, $\operatorname{tub}(r) \subset \operatorname{im}\phi$.

(vi) Using part (iii) one finds

$$\begin{aligned}
\frac{\partial\phi}{\partial s}(s,\alpha) &= \gamma'(s) + r\cos\alpha\, N'(s) + r\sin\alpha\, B'(s)\\
&= T(s) + r\cos\alpha\big(-\kappa(s)\,T(s) + \tau(s)\,B(s)\big) - r\sin\alpha\,\tau(s)\,N(s)\\
&= (1-r\kappa(s)\cos\alpha)T(s) - r\tau(s)\sin\alpha\, N(s) + r\tau(s)\cos\alpha\, B(s),\\
\frac{\partial\phi}{\partial\alpha}(s,\alpha) &= -r\sin\alpha\, N(s) + r\cos\alpha\, B(s),\\
\frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha) &= -r\sin\alpha\big(1-r\kappa(s)\cos\alpha\big)B(s) - r\cos\alpha\big(1-r\kappa(s)\cos\alpha\big)N(s)\\
&\qquad -r^2\tau(s)\sin\alpha\,\cos\alpha\, T(s) + r^2\tau(s)\sin\alpha\,\cos\alpha\, T(s)\\
&= -r\big(1-r\kappa(s)\cos\alpha\big)\big(\cos\alpha\, N(s) + \sin\alpha\, B(s)\big),\\
\left\|\frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha)\right\| &= r(1-r\kappa(s)\cos\alpha).
\end{aligned}$$

(vii) In view of the preceding part

$$r\,\kappa(s) < 1 \quad\Longrightarrow\quad 1-r\kappa(s)\cos\alpha > 0 \quad\Longrightarrow\quad \frac{\partial\phi}{\partial s}\times\frac{\partial\phi}{\partial\alpha}(s,\alpha) \neq 0,$$

which implies that $\frac{\partial\phi}{\partial s}(s,\alpha)$ and $\frac{\partial\phi}{\partial\alpha}(s,\alpha)$ are linearly independent, that is $\operatorname{rank} D\phi(s,\alpha) = 2$, in other words, ϕ is an immersion. The second assertion is the Immersion Theorem 4.3.1.(i).

(viii) Consider $x \in \operatorname{tub}(r)$. According to part (ii) we have $x \in \perp(s)$, for a unique $s \in J$. Hence $x \in \operatorname{tub}(r, s)$, and by part (v) this implies $x = \phi(s, \alpha)$, for a unique $\alpha \in \,]-\pi, \pi\,]$; hence ϕ is injective. Next, suppose $x = \phi(s, \alpha)$. Then

$$\langle\, x - \gamma(s),\, N(s)\,\rangle = r\cos\alpha \qquad \text{and} \qquad \langle\, x - \gamma(s),\, B(s)\,\rangle = r\sin\alpha$$

yield that α depends continuously on x. As $\gamma(s) = x - r\cos\alpha\, N(s) - r\sin\alpha\, B(s)$, it follows that $\gamma(s)$ depends continuously on x; and so s itself too, because γ is an embedding. This proves that ϕ is an embedding.

(ix) As ϕ is an embedding one obtains from (v)

$$\begin{aligned} \operatorname{area}_2((\operatorname{tub}(r)) &= \int_{J\times\,]-\pi,\,\pi\,]} \Big\| \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial\alpha}(s,\alpha) \Big\|\, d(s,\alpha) \\ &= \int_J \Big(\int_{-\pi}^{\pi} r(1 - r\kappa(s)\cos\alpha)\, d\alpha \Big)\, dr = 2\pi r \operatorname{length}(\gamma). \end{aligned}$$

(x) Part (ix) implies directly

$$\operatorname{vol}_3(\operatorname{Tub}(r)) = \int_0^r \operatorname{area}_2(\operatorname{tub}(r))\, dr = \pi r^2 \operatorname{length}(\gamma).$$

(xi) Since $\frac{\partial\Psi}{\partial t_1} \times \frac{\partial\Psi}{\partial t_2} = N \times B = T$, we only need to know the component of $\frac{\partial\Psi}{\partial s}$ along T for the computation of the inner product. Now we have, applying the formulae of Frenet–Serret once more,

$$\frac{\partial\Psi}{\partial s} = \gamma'(s) + t_1\, N'(s) + t_2\, B'(s) \equiv T - \kappa t_1\, T = (1 - \kappa\, t_1)\, T.$$

(xii) The Change of Variables Theorem 6.6.1 implies

$$\begin{aligned} \operatorname{vol}_3(U) &= \int_U dx = \int_{\bigcup_{s\in J}(\{s\}\times D(s))} (1 - \kappa(s)\, t_1)\, d(s,t) \\ &= \int_J \Big(\int_{D(s)} (1 - \kappa(s)\, t_1)\, dt\, ds = \int_J \Big(\operatorname{area}(D(s)) - \kappa(s) \int_{D(s)} t_1\, dt \Big)\, ds \\ &= \int_{\operatorname{im}(\gamma)} \Big(a(s) - \kappa(s) \int_{U(s)} \langle\, y - \gamma(s),\, N(s)\,\rangle\, d_2 y \Big)\, d_1 s. \end{aligned}$$

The last equality follows upon noting that $t \mapsto (\langle\, \Psi(t) - \gamma(s),\, N(s)\,\rangle, \langle\, \Psi(t) - \gamma(s),\, B(s)\,\rangle)$ is the identity mapping in $\mathbf{R}^2$.

Introduce the *moments* $m_B(s)$ and $m_N(s)$ of the planar set $U(s)$ about the lines $\gamma(s) + \mathbf{R}\, B(s)$ and $\gamma(s) + \mathbf{R}\, N(s) \subset \perp(s)$, respectively, by

$$m_B(s) = \int_{U(s)} \langle\, y - \gamma(s),\, N(s)\,\rangle\, d_2 y \qquad \text{and} \qquad m_N(s) = \int_{U(s)} \langle\, y - \gamma(s),\, B(s)\,\rangle\, d_2 y.$$

Then the *centroid* $c(s) \in \perp(s)$ of $U(s)$ with respect to $\gamma(s)$ is defined by

$$c(s) = \frac{1}{a(s)}(m_B(s),\, m_N(s)).$$

These definitions then lead to the formulae

$$\operatorname{vol}_3(U) = \int_{\operatorname{im}(\gamma)} \big(a(s) - \kappa(s)\, m_B(s) \big)\, d_1 s = \int_{\operatorname{im}(\gamma)} a(s)(1 - \kappa(s)\, c_1(s))\, d_1 s.$$

(xiii) Note that the helix is parametrized by arc length and that $\kappa(s) = \frac{1}{2}\sqrt{2}$ for all $s \in J$. Furthermore, Ψ is a diffeomorphism in this case, $a(s) = 1$ and $\int_{D(s)} t_1 \, dt = \int_0^1 t_1 \, dt_1 = \frac{1}{2}$. Thus, the assertion is a direct application of the formula in the preceding part.