Exercise 0.1 (Adjoints, vector calculus, quaternions and Euclidean Dirac operator). Write C for the linear space $C_c^\infty(\mathbf{R}^3)$ of C^∞ functions on $\mathbf{R}^3$ with compact support and introduce the usual inner product on C by $\langle f, g \rangle_C = \int_{\mathbf{R}^3} f(x)g(x)\,dx$, for f and $g \in C$. Consider the linear operator $D_j : C \to C$ of partial differentiation with respect to the j-th variable, for $1 \leq j \leq 3$.

(i) Prove that D_j is anti-adjoint with respect to the inner product on C, that is,

$$\langle D_j f,\, g \rangle_C = -\langle f,\, D_j g \rangle_C.$$

Denote by V the linear space of C^∞ vector fields on $\mathbf{R}^3$ with compact support and introduce an inner product on V by $\langle v,\, w \rangle_V = \int_{\mathbf{R}^3} \langle v(x),\, w(x) \rangle\,dx$, for v and $w \in V$. Here the inner product at the right-hand side is the usual inner product of vectors in $\mathbf{R}^3$. Furthermore, consider the linear operators $\mathrm{grad} : C \to V$ and $\mathrm{div} : V \to C$.

(ii) For $f \in C$ and $v \in V$, verify the following identity of functions in C:

$$\mathrm{div}(f\,v) = \langle \mathrm{grad}\, f,\, v \rangle + f\, \mathrm{div}\, v.$$

Use this to prove

$$\langle \mathrm{grad}\, f,\, v \rangle_V = -\langle f,\, \mathrm{div}\, v \rangle_C.$$

Conclude that $-\mathrm{div} : V \to C$ is the adjoint operator of $\mathrm{grad} : C \to V$.

(iii) For v and w in V, prove the following identity of functions in C:

$$\mathrm{div}(v \times w) = \langle \mathrm{curl}\, v,\, w \rangle - \langle v,\, \mathrm{curl}\, w \rangle.$$

Hint: At the left-hand side the operator D_1 only occurs in the term $D_1(v_2 w_3 - v_3 w_2)$ and apply Leibniz' rule. Next determine the occurrence of D_1 at the right-hand side.

(iv) Deduce from part (iii) that

$$\langle \mathrm{curl}\, v,\, w \rangle_V = \langle v,\, \mathrm{curl}\, w \rangle_V.$$

In other words, the linear operator $\mathrm{curl} : V \to V$ is self-adjoint.

Now consider the following matrix of differentiations acting on mappings $\begin{pmatrix} v \\ f \end{pmatrix} : \mathbf{R}^3 \to \mathbf{R}^4$:

$$M = \begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\mathrm{div} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -D_3 & D_2 & D_1 \\ D_3 & 0 & -D_1 & D_2 \\ -D_2 & D_1 & 0 & D_3 \\ -D_1 & -D_2 & -D_3 & 0 \end{pmatrix}.$$

The preceding results (in particular, part (i)) imply that M is a symmetric matrix, which **in this context** must be phrased as $M^t = -M$ (when "truly" transposing the matrix we also have to take the transpose of its coefficients).

(v) Verify that $-M^2$ equals Gram's matrix associated to M, that is, the matrix containing the inner products of the column vectors of M. Deduce $M^2 = -\Delta E$, where Δ is the Laplacian and E the 4×4 identity matrix. Derive, for $f \in C$ and $v \in V$

$$\mathrm{curl}\,\mathrm{grad}\, f = 0, \qquad \mathrm{div}\,\mathrm{curl}\, v = 0, \qquad \mathrm{curl}(\mathrm{curl}\, v) = \mathrm{grad}(\mathrm{div}\, v) - \Delta v,$$

where in the third identity the Laplacian Δ acts by components on v. Finally, show how to derive the second identity from the first.

Background. We may write $M = I\,D_1 + J\,D_2 + K\,D_3$, where the antisymmetric and orthogonal matrices

$$I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

in $\mathrm{Mat}(4, \mathbf{R})$ satisfy $I^2 = J^2 = K^2 = IJK = -E$. As a consequence $IJ = -JI = K$. Phrased differently, the linear space over $\mathbf{R}$ spanned by E, I, J, K provided with these rules of multiplication forms the noncommutative field $\mathbf{H}$ of the *quaternions*, see the Background in Exercise 5.67. In addition, analogously to the situation in dimension 1 where $-D_1^2 = (i\,D_1)^2$, we have decomposed minus the Laplacian on $\mathbf{R}^3$ into a square of a matrix-valued linear differential operator acting on $\mathbf{R}^4$:

$$-(D_1^2 + D_2^2 + D_3^2)E = (I\,D_1 + J\,D_2 + K\,D_3)^2.$$

M is called the *Euclidean Dirac operator*, which is studied in the theory of *Clifford algebras*.

Solution of Exercise 0.1

(i) Because f and g are of compact support, it is possible to select an open ball $\Omega \subset \mathbf{R}^n$ containing $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$; in particular, f and g vanish along $\partial\Omega$. The formula then follows from Corollary 7.6.2 because the integral over $\partial\Omega$ vanishes.

(ii) On account of Leibniz' rule we have

$$\mathrm{div}(f\,v) = \sum_{1\le j\le 3} D_j(f\,v_j) = \sum_{1\le j\le 3} (D_j f)\,v_j + \sum_{1\le j\le 3} f\,D_j v_j = \langle\, \mathrm{grad}\, f,\; v\,\rangle + f\,\mathrm{div}\,v.$$

Next integrate this identity over $\mathbf{R}^3$ and notice that Gauss' Divergence Theorem 7.8.5 implies that the integral of the left-hand side equals $\int_{\partial\Omega} f(y)\langle\, v(y),\, \nu(y)\,\rangle\,dy = 0$, for the same reasons as in part (i). The final conclusion is a consequence of the definition of the adjoint in Section 2.1.

(iii) At the left-hand side D_1 occurs in the term $v_2 D_1 w_3 + w_3 D_1 v_2 - v_3 D_1 w_2 - w_2 D_1 v_3$, while at the right-hand side it occurs in $-w_2 D_1 v_3 + w_3 D_1 v_2 + v_2 D_1 w_3 - v_3 D_1 w_2$, which is a rearrangement of the former expression. Taking the indices modulo 3 one obtains analogous results for D_2 and D_3 by means of cyclic permutation of the indices.

(iv) The desired results follow in the same manner as in part (ii).

(v) First note that $-M^2 = M^t M$ where the right-hand side is Gram's matrix according to Section 2.1. On the basis of the symmetry of Gram's matrix and $D_i D_j = D_j D_i$, one has to perform 10 trivial mental calculations to establish that $\langle\, M_i,\, M_j\,\rangle = \delta_{ij}\,\Delta$, for $1 \le i, j \le 3$. This leads to $M^2 = -\Delta E$. On the other hand one finds

$$M^2 = \begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\mathrm{div} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{curl} & \mathrm{grad} \\ -\mathrm{div} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{curl}\circ\mathrm{curl} - \mathrm{grad}\circ\mathrm{div} & \mathrm{curl}\circ\mathrm{grad} \\ -\mathrm{div}\circ\mathrm{curl} & -\mathrm{div}\circ\mathrm{grad} \end{pmatrix}.$$

Comparison of the matrix coefficients leads to the desired conclusions. Observe that in addition one recovers the definition $\Delta = \mathrm{div}\circ\mathrm{grad}$. The second identity follows from the first by taking the transpose.